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Bridges in Highly Connected Graphs

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Abstract

Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be a set of internally disjoint paths contained in a graph G and let S be the subgraph defined by $\bigcup_{i=1}^{t} P_i$. A *bridge* of S is either an edge of $G - E(S)$ with both endpoints in $V(S)$ or a component C of $G - V(S)$ along with all the edges from $V(C)$ to $V(S)$. The attachments of a bridge B are the vertices of $V(B) \cap V(S)$. A bridge B is k-stable if there does not exist a subset of at most $k-1$ paths in $\mathcal{P}$ containing every attachment of B. A theorem of Tutte states that if G is a 3-connected graph, there exist internally disjoint paths $P_1', \ldots, P_t'$ such that P_i and P_i' have the same endpoints for $1 \leq i \leq t$ and every bridge is 2-stable. We prove that if the graph is sufficiently connected, the paths $P_1', \ldots, P_t'$ may be chosen so that every bridge containing at least two edges is in fact k-stable. We also give several simple applications of this theorem to a conjecture of Lovász on deleting paths maintaining high connectivity.

Key Words : graph connectivity, graph bridges, non-separating cycles

1 Introduction

Let S be a subgraph of a graph G. An *S-bridge* in G is a connected subgraph B of G such that $E(B) \cap E(S) = \emptyset$ and either $E(B)$ is an edge with both endpoints contained in $V(S)$ or for some component C of $G - V(S)$ the set $E(B)$ consists of all edges of G with at least one endpoint in C. The vertices of $V(B) \cap V(S)$ are the *attachments* of B. A bridge is *trivial* if it consists of a single edge and non-trivial otherwise. The set of *branch vertices of* S is any subset $X \subseteq V(S)$ such that X contains every vertex of S of deg_S at least three. A *segment* of S is a non-trivial path P contained in S with both endpoints equal to a branch vertex and no branch vertex contained as an internal vertex of P. Observe by definition that the segments are internally disjoint subpaths of S. For a given subgraph S and fixed set X of branch vertices, we will typically refer to the segments of S to mean the collection of all segments of S with branch set X.

Let S be a subgraph of a graph G with branch vertices X. For any integer $k \geq 1$, an S-bridge B is *k-stable* if there do not exist a subset of at most $k-1$ segments of S containing every attachment of B. The following theorem is due to Tutte [13]:

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Theorem 1.1 (Tutte) *Let S be a subgraph of a 3-connected graph G and let $P_1, \ldots, P_t$ be the segments of S. Then there exists a subgraph S' with segments $P'_1, \ldots, P'_t$ such that P_i and P'_i have the same endpoints for $1 \leq i \leq t$. Moreover, every S'-bridge B is 2-stable.*

Theorem 1.1 essentially says that given any three connected graph G and a collection of internally disjoint paths $P_1, \ldots, P_t$, it is possible to reroute the paths $P_1, \ldots, P_t$ to get internally disjoint paths $P'_1, \ldots, P'_t$ such that $P'_1, \ldots, P'_t$ preserve the endpoints and moreover, every bridge attaches to at least two distinct paths of $P'_1, \ldots, P'_t$. The main theorem of this article is the following extension of Tutte's theorem to force bridges having arbitrarily large stability. Given a subgraph of a graph G with segments $P_1, P_2, \ldots, P_t$, a subset of the segments $\{P_i : i \in I\}$ for some set $I \subseteq \{1, 2, \ldots, t\}$ is *independent* if for every index $j \in I$, $V(P_j) \nsubseteq \bigcup_{i \in I, i \neq j} V(P_i)$.

Theorem 1.2 *Let $k \geq 1$ be given and let G be a $243k$-connected graph. Let S be a subgraph of G with branch vertices X and segments $P_1, \ldots, P_t$ such that there exists an independent set of k segments. Furthermore, assume for any non-trivial segment P_i with endpoints x and y, the graph $G - E(S)$ does not contain the edge xy. Then there exists a subgraph S' with branch vertices X and segments $P'_1, \ldots, P'_t$ such that P'_i has the same endpoints as P_i for $1 \leq i \leq t$ such that every trivial S'-bridge has endpoints on two distinct segments, and every non-trivial S'-bridge is k-stable. Also, if there exists a non-trivial bridge of S, then there exists a non-trivial bridge of S'.*

The statement of Theorem 1.2 is made more complex by the fact that we allow the segments of S to consist of both single edges as well as paths of length two or more. When the segments are all assumed to be induced paths of length at least two, the statement is simpler.

Corollary 1.3 *Let $k \geq 1$ be given and let G be a $243k$-connected graph. Let S be a subgraph of G with branch vertices X and segments $P_1, \ldots, P_t$ where $t \geq k$ and P_i is an induced path of length at least two for all $1 \leq i \leq t$. Then there exists a subgraph S' with branch vertices X and segments $P'_1, \ldots, P'_t$ such that P'_i is an induced path with the same endpoints as P_i for $1 \leq i \leq t$ and furthermore, every non-trivial S'-bridge is k-stable.*

We will make use of the following notation. Where not otherwise stated, we follow the notation of [2]. A *separation* (A, B) of a graph G is a pair of subsets of $V(G)$ such that $A \cup B = V(G)$ and every edge xy is either contained in the subgraph of G induced by A or the subgraph induced by B. A separation (A, B) is trivial if $A \subseteq B$ or $B \subseteq A$. Given a path P in a graph G and two specified vertices x and y in P, we refer to the subpath of P with endpoints x and y by xPy. The proof of Theorem 1.2 as well as the applications of Theorem 1.2 will make use of the theory of graph linkages. A *linkage problem of size k* in a graph G is a multiset of k subsets of $V(G)$ of size two $\mathcal{L} = \{\{s_i, t_i\} : s_i, t_i \in V(G), 1 \leq i \leq k\}$. A *solution* to a given linkage problem $\mathcal{L} = \{\{s_i, t_i\} : s_i, t_i \in V(G), 1 \leq i \leq k\}$ is a set of k internally disjoint paths $P_1, P_2, \ldots, P_k$ where the endpoints of P_i are s_i and t_i for all $1 \leq i \leq k$. A graph G is *k-linked* if it has at least $2k$ vertices and if there exists a solution to every linkage problem of size k with pairwise disjoint subsets of size two. By assuming a sufficient amount of connectivity, we can assume a given graph is k-linked.

Theorem 1.4 ([11]) *Every $10k$-connected graph is k-linked.*

A graph G is *strongly k-linked* if every linkage problem of size k has a solution. It has been independently shown by Mader [10] as well as by Liu, West, and Yu [7] that every k-linked graph on at least $2k$ vertices is also strongly k-linked. This immediately implies following corollary to Theorem 1.4.

Corollary 1.5 *Every $10k$-connected graph is strongly k-linked.*

Corollary 1.5 can also be directly proven from Theorem 1.4 by a simple construction duplicating vertices contained in multiple pairs of a given linkage problem.

Before giving the proof of Theorem 1.2, we first examine several implications of the theorem to problems arising from a collection of questions we will generally refer to as removable paths conjectures. The following conjecture is due to Lovász:

Conjecture 1.6 *There exists a function $f(k)$ such that for every $f(k)$-connected graph G and every pair of vertices s and t of G, there exists an s-t path P such that $G - V(P)$ is k-connected.*

Even the existence of such a function $f(k)$ remains open, leading to the study of a variety of weaker versions of the conjecture. In Section 2, we present two applications of Theorem 1.2 to questions related to Conjecture 1.6. In the following sections, we give the proof of Theorem 1.2. We conclude with a brief discussion on the possibility of improving the amount of connectivity necessary in Theorem 1.2 as well as a lower bound for the best possible value of the constant.

2 Applications to removable path questions

Conjecture 1.6 has been shown to be true for small values of k. The case when $k = 1$ is an immediate consequence of Theorem 1.1. To see this, let G be a 3-connected graph and let u and v be any pair of vertices of G. By Theorem 1.1 there exist two non-trivial paths linking u and v such that every non-trivial bridge is 2-stable. Then either of the paths can be deleted and leave the remaining graph connected. A path P in a graph G where $G - V(P)$ is connected is called a *non-separating* path. The $k = 1$ case of Conjecture 1.6 can be rephrased to state that there exists a non-separating path connecting any pair of vertices, assuming the graph satisfies some connectivity bound. Chen, Gould, and Yu [1] show in fact that a highly connected graph contains many internally disjoint non-separating paths linking any pair of vertices.

Theorem 2.1 ([1]) *Let k be a positive integer and let G be a $(22k + 2)$-connected graph. Then for any pair of vertices u and v of G there exist k internally disjoint non-separating paths $P_1, P_2, \ldots, P_k$ such that the endpoints of P_i are u and v for every $1 \leq i \leq k$.*

A set of internally disjoint paths $\{P_1, P_2, \ldots, P_k\}$ contained in a graph G is *batch non-separating* if for any subset $I \subseteq \{1, 2, \ldots, k\}$ the graph $G - \left(\bigcup_{i\in I} V(P_i)\right)$ is connected. A consequence of Theorem 1.2 is that a highly connected graph in terms of k contains a batch non-separating set of internally disjoint paths connecting any pair of vertices.

Theorem 2.2 *Let H be a multigraph without loops and with no isolated vertices. Let $l = |E(H)|$. Let G be a $243(l + 1)$-connected graph. Then for any injective function $\rho : V(H) \to V(G)$ there exist internally disjoint paths $\{P_{xy} : xy \in E(H)\}$ such that the endpoints of P_{xy} are $\rho(x)$ and $\rho(y)$ for all edges $xy \in E(H)$ and furthermore, the set of paths $\{P_{xy} : xy \in E(H)\}$ is batch non-separating.*

When H is assumed to be the multigraph consisting of k parallel edges connecting two vertices, Theorem 2.2 strengthens Theorem 2.1 to find a set of batch non-separating paths linking any pair of vertices, albeit with a worse constant. We note that the property ensuring the existence of internally disjoint paths $\{P_{xy} : xy \in E(H)\}$ for every function $\rho : V(H) \to V(G)$ is known as H-linked and has recently been studied. See [3][4] for more details.

Proof. **[Theorem 2.2]** Let G and H be given with $l = |E(H)|$. Assume that G is $243(l+1)$-connected and let $\rho : V(H) \to V(G)$ be a fixed injective map. Let s and t be distinct vertices of G disjoint from $\{\rho(v) : v \in V(H)\}$. Set X to be the set of vertices $\{\rho(v) : v \in V(H)\} \cup \{s,t\}$. Consider the linkage problem consisting of the multiset $\mathcal{L} = \{\{\rho(x), \rho(y)\} : xy \in E(H)\} \cup \{s,t\}$. There are two technicalities here to deal with. First, when we apply Theorem 1.2, we will need the set of paths to be an independent set of paths of size $l+1$, and, second, we want to ensure there exists at least one non-trivial bridge. Thus, we set $E_X = \{xy \in E(G) : \{x,y\} \in \mathcal{L}\}$ and fix a vertex v in $V(G) - X$. Now we find a solution $P_1, P_2, \ldots, P_l, P_{l+1}$ to the linkage problem $\mathcal{L}$ in the graph $(G - E_X) - \{v\}$. Such a solution exists because the connectivity of $(G - E_X) - \{v\}$ is at least $243(l+1) - l - 1 \geq 10(l+1)$.

Let S be the subgraph of G consisting of $\left(\bigcup_{i=1}^{l+1} P_i\right) \cup E_X$. Then with branch set X the segments of S are the edges of E_X and the paths $P_1, P_2, \ldots, P_{l+1}$. The set of paths $P_1, P_2, \ldots, P_{l+1}$ is an independent set of size $l+1$, so by Theorem 1.2, there exists a subgraph S' with branch set X such that every non-trivial bridge is $(l+1)$-stable. Moreover, since S does not contain the vertex v by construction, there exists at least one non-trivial bridge of S'. Observe that such a non-trivial bridge must have an internal vertex of every segment of S' of length at least two as an attachment. For every edge xy of H, we let P'_{xy} either be the edge $\rho(x)\rho(y)$ if $\rho(x)$ and $\rho(y)$ are adjacent in G and otherwise, we let P'_{xy} be the segment of S' with ends $\rho(x)$ and $\rho(y)$. We claim $\{P'_{xy} : xy \in E(H)\}$ is a batch non-separating set of paths satisfying the statement of the theorem. To see this, let P be the segment of S' linking s and t. Every non-trivial bridge has an attachment in P. Also, every segment of S' of length at least two has an internal vertex attaching to a $(l+1)$-stable bridge. Thus, upon deleting any subset of the segments $\{P'_{xy} : xy \in E(H)\}$, any remaining vertex has a path to P, implying that the remaining graph is connected. This completes the proof of the theorem. ∎

The case of Conjecture 1.6 when $k = 2$ has been shown to be true independently by Kriesell [6] and Chen et al [1] where they show that every 5-connected graph contains a path linking any two vertices such that deleting the path leaves the remaining graph 2-connected. The first open case is when $k = 3$. The following theorem is due to Kawarabayashi, Reed, and Thomassen [5].

Theorem 2.3 ([5]) *There exists a constant c such that for every c-connected graph G and every pair of vertices s and t of G, there exists an s-t path P and a 3-connected subgraph H such that such that $G - V(P)$ is isomorphic to a subdivision of H.*

Kawarabayashi et al prove Theorem 2.3 from first principles. We give the following short proof using Theorem 1.2.

Proof. (Theorem 2.3) Let $c = 729$, and let G be a c-connected graph. Fix the vertices s and t in G. Let H be a 3-connected graph and P an s-t path in G such that $G - P$ contains a subgraph isomorphic to a subdivision of H. Such a graph H exists, since by Corollary 1.5, G is strongly 7-linked and we can find a subdivision of K_4 disjoint from a path linking s and t. Assume such a subgraph H and path P are chosen to maximize $|V(H)| + |E(H)|$. Let S_H be the subgraph of $G - P$ isomorphic to a subdivision of H, and let $X := \{v \in V(S_H) : deg_{S_H}(v) \geq 3\} \cup \{s,t\}$ be the set of vertices of S_H of degree at least three and the vertices s and t. Moreover, we may assume that every segment of S_H is an induced path. Since X contains at least four branch vertices from S_H disjoint from the path P, we see that $S_H \cup P$ with branch set X contains a set of 3 independent segments. Thus, by our choice of connectivity of G, we may assume that every non-trivial bridge of $S_H \cup P$ is 3-stable and maintain the property that every segment is induced. Let Q be a segment of S_H of length at least two. Every bridge attaching to an internal vertex of Q can only have attachments on $Q \cup P$. Otherwise, there exists a graph H' obtained from H either by subdividing one or two edges of H and adding an edge, such that a subdivision of H' is contained as a subgraph of $G - P$. This contradicts our choice of H. Given that every non-trivial bridge is 3-stable, it follows that the only bridges attaching to internal vertices of Q are single edges with one end in Q and one end in P. Finally, we see that there is no non-trivial bridge of $S_H \cup P$. If such a bridge exists, it must have as attachment either two vertices of $X \cap V(S_H)$ that are not connected by a segment, or the bridge must attach to three vertices of $X \cap V(S_H)$. In the first case, there exists a graph H' obtained from H by adding an edge such that $G - P$ contains a subdivision of H'. In the second case, there exists a graph H' obtained from H by adding a new vertex with three neighbors in $V(H)$ such that $G - P$ contains a subdivision of H'. In either case, we contradict our choice of H. Thus we see that $G - P$ is exactly the graph S_H, proving the theorem. ■

3 Finding a Comb

We recall that a *linkage* is a graph where every component is a path. Given two sets X and Y in a graph G, a linkage $\mathcal{Q}$ contained in G is a linkage from X to Y if every component of $\mathcal{Q}$ has one endpoint in X and one endpoint in Y and is otherwise disjoint from $X \cup Y$. In a slight abuse of notation, we will often use $P \in \mathcal{P}$ to refer to a component P in a linkage $\mathcal{P}$.

Definition Let G be a graph and S a subgraph of G with branch vertices X and segments $P_1, \ldots, P_t$. Let k be an integer. Let H be a subgraph of G and let $\mathcal{Q}$ be a linkage from $V(H)$ to $V(S)$ with k components $Q_1, Q_2, \ldots, Q_k$. Let P be a segment of S with endpoints x and x'. A vertex $v \in V(P) \cap V(\mathcal{Q})$ is *extremal* if v is the unique vertex of $V(\mathcal{Q})$ in the subpath xPv or in the subpath vPx'. A vertex $v \in V(P)$ is *$\mathcal{Q}$-sheltered* by the extremal vertices y and y' in $V(P)$ if v is contained in the subpath yPy'. The linkage $\mathcal{Q}$ forms a *H-comb* if the following hold:

1. For each $Q \in \mathcal{Q}$, one endpoint is contained in $V(H)$ and the other endpoint is a $\mathcal{Q}$-extremal vertex.

2. Every $\mathcal{Q}$-extremal vertex is the terminus of some component of $\mathcal{Q}$.

3. If some vertex of $V(H) \cap V(P)$ for some segment P of S is not a terminus of any path $Q \in \mathcal{Q}$, and it is not $\mathcal{Q}$- sheltered, then every path of $\mathcal{Q}$ has length zero and P includes the terminus of at most one path in $\mathcal{Q}$.

The main result of this section will be to provide a necessary and sufficient condition for the existence of a H-comb in a given graph G. Towards that end, we will prove the following lemma. We first define the following. Let G be a graph and let S and H be two subgraphs. Let X be the branch set of S. A linkage $\mathcal{P}$ with components $P_1, P_2, \ldots, P_t$ from $V(H)$ to $V(S)$ is *extremal* if for every index i, the endpoint of the component P_i of $\mathcal{P}$ in $V(H)$ is a $\mathcal{P}$-extremal vertex. A *truncation* of $\mathcal{P}$ is a linkage with t components where every component is of the form $x_i P_i z$ where $x_i \in V(H)$ and $z \in V(S)$.

Lemma 3.1 *Let G be a graph and S a subgraph with branch set X. Let H be a subgraph and let $\mathcal{Q}$ be a linkage from $V(H)$ to $V(S)$. If $\mathcal{Q}$ is extremal, then there exists a truncation $\mathcal{Q}'$ of $\mathcal{Q}$ such that $\mathcal{Q}'$ is both extremal and every $\mathcal{Q}'$-extremal vertex is the endpoint of a component of $\mathcal{Q}'$.*

Proof. Let G, S, H, and the linkage $\mathcal{Q}$ with components $Q_1, Q_2, \ldots, Q_k$ be given. Let the endpoint of Q_i in $V(H)$ be labeled x_i for all $1 \leq i \leq k$. Let $\mathcal{Q}'$ be an extremal truncation of $\mathcal{Q}$ containing a minimal number of vertices. Observe that $\mathcal{Q}$ itself is trivially an extremal truncation of $\mathcal{Q}$. We claim $\mathcal{Q}'$ is as desired in the statement of the lemma. Let the components of $\mathcal{Q}'$ be $Q_1', Q_2', \ldots, Q_k'$ with the path $Q_i' = x_i Q_i w_i$ for some vertex $w_i \in V(S)$. Assume to reach a contradiction, that there exists a vertex $z \in Q_j'$ for some index $1 \leq j \leq k$ such that z is $\mathcal{Q}'$-extremal but that z is not the endpoint of any component of $\mathcal{Q}'$. Consider the truncation $\overline{\mathcal{Q}}$ of $\mathcal{Q}'$ obtained by deleting the path Q_j and adding the path $x_j Q_j z$, i.e. $\overline{\mathcal{Q}} = (\mathcal{Q}' - Q_j') \cup x_j Q_j z$. Every vertex of $\mathcal{Q}'$ that is $\mathcal{Q}'$-extremal is also $\overline{\mathcal{Q}}$-extremal. It follows that $\overline{\mathcal{Q}}$ is an extremal truncation violating our choice of $\mathcal{Q}'$ to contain a minimal number of vertices. ■

The following lemma is a strengthening of Lemma 3.1 of [12].

Lemma 3.2 *Let G be a graph and let S be a subgraph of G with branch set X. Let $k \geq 1$ be given and let H be a subgraph of G. Let $P_1, \ldots, P_t$ be the segments of S. Either there exists a separation (A, B) of order at most $k-1$ (possibly a trivial separation) with $X \subseteq A$ and $V(H) \subseteq B$, or there exists an H-comb with k components.*

Proof. Let G, S, H, X, k, and $P_1, \ldots, P_t$ be given. Assume there does not exist a separation as stated in the lemma. Then there exists a linkage $\mathcal{Q}$ from $V(H)$ to X with k-components. Choose such a linkage $\mathcal{Q}$ such that $E(\mathcal{Q}) - E(S)$ is minimal.

Let $Q_1, \ldots, Q_k$ be the components of $\mathcal{Q}$. For each $i = 1, \ldots, k$, let q_i be the endpoint of Q_i in H. Let $\mathcal{Q}'$ be an extremal truncation of $\mathcal{Q}$ with components $Q_1', Q_2', \ldots, Q_k'$ where for all $1 \leq i \leq k$, the component $Q_i' = q_i Q_i w_i$ for some vertex $w_i \in V(S)$. Such an extremal truncation exists by Lemma 3.1. Assume as well that we pick $\mathcal{Q}'$ to contain a minimum number of vertices. It follows immediately that the linkage $\mathcal{Q}'$ satisfies both properties 1. and 2. in the definition of a H-comb.

To see that $\mathcal{Q}'$ satisfies Condition 3. in the definition of H-comb, assume there exists $x \in V(H) \cap V(P_i)$ for some index i such that x is not $\mathcal{Q}'$-sheltered. Let x_i be an endpoint of P_i such that $x_i P_i x$ does not contain any vertex of $V(\mathcal{Q}')$ other than x. We may also assume that x is the only vertex of $V(H)$ contained in $x_i P_i x$. Since $x \in V(H) - V(\mathcal{Q}')$ and no internal vertex of a component of $\mathcal{Q}$ belongs to H, we deduce that $x \notin V(\mathcal{Q})$. We claim, as well, that $x_i P_i x$ is disjoint from $V(\mathcal{Q})$. Assume otherwise, and let y be a vertex of $\mathcal{Q}$ in $x_i P_i x$ closest to x. Let j be the index such that $y \in V(Q_j)$. By the fact that Q_j does not contain the vertex x and by our choice of x such that $x_i P_i x$ does not contain any other vertex of H, it follows that $y Q_j q_j$ must contain an edge not in $E(S)$. By considering the linkage $(\mathcal{Q} - Q_j) \cup (x P_i y \cup y Q_j w)$ where w is the end of Q_j in X, we obtain a linkage contradicting our choice of $\mathcal{Q}$ to minimize $E(\mathcal{Q}) - E(S)$

We conclude that $x_i P_i x$ is disjoint from $\mathcal{Q}$. The path $x_i P_i x$ could have been chosen for the linkage $\mathcal{Q}$. Again by our choice of $\mathcal{Q}$ to minimize $E(\mathcal{Q}) - E(S)$, it follows that $\mathcal{Q}$ is a subgraph of S. Given that we chose $\mathcal{Q}'$ to contain a minimal number of vertices, it follows that each Q_j' is a trivial path of length zero. Since $x_i P_i x$ is disjoint from $V(\mathcal{Q})$, we conclude that P_i contains the endpoint of at most one path of $\mathcal{Q}'$. Thus Condition 3. in the definition of H-comb holds and the lemma is proven. ■

4 Proof of Theorem 1.2 and a lower bound

The proof of Theorem 1.2 will require the following classic theorem of Mader.

Theorem 4.1 ([9]) *Every graph with minimum degree at least $4k$ contains a k-connected subgraph.*

We now proceed with the proof of Theorem 1.2.

Proof. [Theorem 1.2] Let k be a positive integer and let S be a subgraph of G with branch vertices X and segments $P_1, \dots, P_t$ with $t \geq k$ as in the statement of the theorem. We will modify our set of branch vertices several times in the proof. To avoid confusion, for the remainder of the proof we will refer to the segments, bridges, and combs as defined for a given ordered pair (H, Y) for some subgraph H and branch set Y of vertices in $V(H)$. Two subgraphs H and H' both with branch set X are *segment equivalent* if for every segment P of H, there exists a segment Q of H' such that P and Q have the same endpoints and vice versa. Let S' be a subgraph of G with branch set X such that (S', X) and (S, X) are segment equivalent. Let the segments of (S', X) be P_i' for $1 \leq i \leq t$ where the segments P_i and P_i' have the same endpoints. Furthermore, assume we pick S' such that

(i) The number of vertices contained in k-stable bridges is maximized, and

(ii) subject to (i), the number of vertices in $|V(S)|$ is minimized.

Property (ii) has two immediate implications. First, we see that every trivial (S', X)-bridge does not have both endpoints contained in a single segment of (S', X). Assume otherwise and let xy be an edge of $E(G) \setminus E(S')$ such that both x and y are contained in P_j' for a fixed index j. Then if the endpoints of P_j' are u_1 and u_2, we can replace P_j' in S' with the segment $u_1P_j'xyP_j'u_2$ to find a segment equivalent subgraph with fewer vertices than S'. Moreover, by the assumption that no edge u_1u_2 exists in $E(G) - E(S)$, it follows that at least one of x and y is not in $\{u_1, u_2\}$. Thus any k-stable bridge that has an attachment as an internal vertex of $xP_j'y$ still has an internal vertex of $u_1P_j'xyP_j'u_2$ as an attachment, and, consequently, is still k-stable upon rerouting the segment P_j'.

Second, property (ii) implies that if a vertex $v \in V(G) \setminus V(S')$ is not contained in a k-stable (S', X)-bridge, then v has at most $3(k-1)$ neighbors in S'. Otherwise, v would have at least four neighbors in a given segment and it would be possible to shorten the segment by routing through the vertex v while at the same time not decreasing the number of vertices contained in k-stable bridges.

Assume, to reach a contradiction, that there exists some non-trivial (S', X)-bridge that is not k-stable. We will define a new set of branch vertices for S'. Let

$$Y_1 = \{v \in V(S') : \ v \text{ is an attachment of a } k\text{-stable } (S', X)\text{-bridge}\}$$

and

$$Y_2 = \{y \in V(S') : \ \exists \text{ an edge } yx \text{ contained in } S' \text{ with } x \in X \cup Y_1\}.$$

We define

$$\overline{X} = X \cup Y_1 \cup Y_2.$$

Let s be a positive integer and let $\overline{P_1}, \overline{P_2}, \dots, \overline{P_s}$ be the segments of $(S', \overline{X})$. Observe that by our choice of $\overline{X}$, for every vertex x of $\overline{X}$, there exists at most one segment $\overline{P}_i$ containing x with $|V(\overline{P}_i)| \geq 3$. We let $l = \max\{6k, |\overline{X}|\}$.

Let G' be the subgraph of G induced by $V(S')$ as well as the vertices of any (S', X)-bridge that is not k-stable. Let B be a non-trivial (S', X)-bridge of G'. Since B is also a (S', X)-bridge in G, we see that for any vertex v of $V(B) \setminus V(S')$, v has at most $3(k-1)$ neighbors in $V(S')$. It follows from Theorem 4.1 that $G[V(B) \setminus V(S)]$ contains a $60k$-connected subgraph H. If we consider the graph G', there cannot be a small separation of order strictly less than l separating $\overline{X}$ from $V(H)$; otherwise such a small separation would extend to a non-trivial separation of G of order l, a contradiction. By Lemma 3.2, in the graph G' there exists an H-comb $\mathcal{Q} = Q_1 \cup Q_2 \cup \dots \cup Q_l$ of order l to the segments of $(S', \overline{X})$. Let q_i be the end of Q_i in H and x_i the end of Q_i in S' for $1 \leq i \leq l$. For all indices

i where x_i is contained in a segment of length at least two, let $\pi(i)$ be the index such that $\overline{P}_{\pi(i)}$ contains the endpoint x_i of Q_i and $|V(\overline{P}_{\pi(i)})| \geq 3$. As we have already observed, there is at most one such segment and so the function π is well defined. We fix a non-negative integer l' and number the components of $\mathcal{Q}$ such that the endpoints x_{2i-1} and x_{2i} of Q_{2i-1} and Q_{2i} respectively are contained in the same segment $\overline{P}_{\pi(2i-1)} = \overline{P}_{\pi(2i)}$ for $1 \leq i \leq l'$, and that for i such that $2l'+1 \leq i \leq l$, either every segment containing x_i consists of a single edge, or the endpoint x_i of Q_i is the unique extremal vertex of $\mathcal{Q}$ contained in $P_{\pi(i)}$.

Using the fact that H is highly connected and the comb linking S' into H, we will reroute the segments of length at least two of (S', X) to increase the number of vertices contained in k-stable bridges, contradicting our choice of S'. Let v and $u_1, u_2, \dots, u_{l'}$ be $l'+1$ distinct vertices of H such that $\{v, u_1, \dots, u_{l'}\} \cap \{q_i : 1 \leq i \leq l\} = \emptyset$. This is possible, given the minimum degree of H is at least $60k$. We find a solution to the linkage problem $\{\{q_{2i-1}, u_i\}, \{q_{2i}, u_i\} : 1 \leq i \leq l'\} \cup \{\{v, q_i\} : 2l'+1 \leq i \leq l\}$. Such a solution exists by Corollary 1.5 and the fact that H is $60k$-connected. For $1 \leq i \leq l'$, let the path linking q_{2i-1} to q_{2i} going through the vertex u_i be labeled R_i. Consider the subgraph T obtained from S' by deleting the interior vertices of $P_{\pi(2i-1)}$ for each $1 \leq i \leq l'$ and replacing it with the path $P_{\pi(2i-1)} x_{2i-1} Q_{2i-1} q_{2i-1} R_i q_{2i} Q_{2i} x_{2i} P_{\pi(2i)}$. The subgraph T contains the vertex set $\overline{X}$ and consequently, the set X is also contained in $V(T)$. Moreover, by construction, the set X contains every vertex of T of deg_T at least 3. If we consider T as a subgraph of G, it follows that (S, X) and (T, X) are segment equivalent. Moreover, because the construction of T fixed the set Y_1 as branch vertices, every k-stable non-trivial bridge of (S', X) in G is also a k-stable bridge of (T, X) in the graph G. Thus to prove the claim, it suffices to show that (T, X) has at least one k-stable bridge in G' and contradict our choice of S' to maximize the number of vertices contained in k-stable bridges.

We focus on the bridge B of $(T, \overline{X})$ in G containing the vertex v. The bridge B has as attachments the vertices $\{u_i : 1 \leq i \leq l'\} \cup \{q_i : 2l'+1 \leq i \leq l\}$. Observe that the vertices u_i are contained as internal vertices of segments of $(T, \overline{X})$. It follows that $l = 6k$. Otherwise, our comb has a path terminating at every vertex of $\overline{X}$, implying that the bridge B attaches to every vertex of X as well as having an attachment contained as an internal vertex of every segment of (T, X). Given that S contains an independent set of segments of size k, it follows that B is k-stable. If $l' \geq k$, it follows that the bridge containing v has as attachments an internal vertex of k distinct segments of $(T, X \cup Y_1)$, since every segment of $(T, X \cup Y_1)$ consists of at most three segments of $(T, \overline{X})$ only one of which can be a path of length two or more. If B is a k-stable bridge of (T, X), then T contradicts our choice of S' to maximize the number of vertices contained in good bridges. We may then assume that there exists some segment L of (T, X) such that L contains two distinct segments L_1 and L_2 of $(T, X \cup Y_1)$ and both L_1 and L_2 contain as an internal vertex an attachment of B. Let the endpoints of L be x and y and let the internal attachments of B in L_i be z_i for $i = 1, 2$. The bridge B contains a path $\overline{L}$ from z_1 to z_2 and otherwise disjoint from T. Consider the subgraph $\overline{T} = T - (L - \{x, y\}) \cup x L z_1 \overline{L} z_2 L y$ obtained from T by rerouting the segment L along the path $\overline{L}$. The fact that L_1 and L_2 are distinct segments of $(T, X \cup Y_1)$ implies that there exists a vertex w of Y_1 contained in L between L_1 and L_2. Specifically, w is contained as an internal vertex of the subpath $z_1 L z_2$. It follows that $(\overline{T}, X)$ has strictly more vertices contained in k-stable bridges than (S', X) since the vertex w that is an attachment of a k-stable bridge of (S', X) is contained in some k-stable bridge of $(\overline{T}, X)$ in G minus its attachments. This contradiction to our choice of S' implies that $l' < k$.

We have reduced to the case when the bridge B of (T, X) has at least $6k - 2l' \geq 4k$ attachments in $X \cup Y_1 \cup Y_2$. Since we may assume B is not a k-stable bridge, there must exist some segment

L of (T, X) containing 5 attachments of B that are also contained in $X \cup Y_1 \cup Y_2$. It follows that there exist vertices z_1, z_2, and w contained in L such that z_i is an attachment of B for $i = 1, 2$, $w \in Y_1 - X$ and w is an internal vertex of $z_1 L z_2$. As in the previous paragraph, by rerouting the segment L through B to avoid the vertex w, we violate our choice of S' by adding the vertex w to the non-attachment vertices of some k-stable bridge.

We conclude that every non-trivial bridge of (S', X) is k-stable. If (S', X) has no non-trivial bridge, then $V(S') = V(G)$. Since in this case it also follows that $|V(S')| \leq |V(S)|$, we conclude that (S, X) has no non-trivial bridge. Conversely, if (S, X) has at least one non-trivial bridge, then (S', X) has at least one k-stable bridge. This completes the proof of the theorem. ■

It would be interesting to know the best possible connectivity function in Theorem 1.2. The large connectivity function obtained in the proof Theorem 1.2 is a consequence of two factors: first, the linkage property used to analyze the highly connected subgraph H, and second, the large number of paths contained in the comb. Possibly by a more careful choice of vertices for $\overline{X}$ and a more detailed analysis of how many comb paths can terminate in the same segment of (S, X) could improve the overall connectivity function in the statement of Theorem 1.2. However, this approach would still likely fail to approach the best known lower bound for the connectivity function of Theorem 1.2.

Consider two large complete graphs G_1 and G_2. Pick two subsets of $2k - 2$ vertices in each and identify them to create a graph G. Let our system of paths in G consist of $k - 1$ disjoint edges in the intersection of the two cliques as well as an additional l disjoint edges from G_1 disjoint from the vertices of G_2. It is impossible to reroute the paths so that every non-trivial bridge is k-stable, as the bridge containing the vertices unique to G_2 will only have attachments in the $k - 1$ paths of the intersection. This example shows that the best possible connectivity function that could be hoped for in Theorem 1.2 would be $2k - 1$. Interestingly, this is the value obtained in the $k = 2$ case in Theorem 1.1 of Tutte.

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