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Packing Steiner Trees on Four Terminals

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Abstract

Let A be a set of vertices of some graph G. An A-*tree* is a subtree of G containing A. An A-*bridge* is a subgraph of G which is either formed by a single edge connecting two vertices of A or by the edges incident with the vertices of some component of $G - A$. An A-bridge is called *binary* if it is a tree and all its vertices outside A have degree 3. A is called k-*edge-connected* in G if every set of less than k edges in G misses at least one A-tree.

We observe first that, for $\ell \in \mathbb{N}$ and any function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) \geq 0$ and $f(k+1) \geq f(k)+1$, the following statements are equivalent: (i) Every $f(k)$-edge-connected set A of ℓ vertices in some graph G admits k edge disjoint A-trees, and: (ii) Every $f(k)$-edge-connected set A of ℓ vertices of degree $f(k)$ in some graph G such that every A-bridge is binary but not an A-tree admits k edge disjoint A-trees. Using this, we prove that every $\lceil \frac{3k}{2} \rceil$-edge-connected set A of four vertices in a graph admits a set of k edge disjoint A-trees. The bound $\lceil \frac{3k}{2} \rceil$ is best possible for all values of k.

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1 Introduction

All graphs considered here are supposed to be finite and undirected but may contain multiple edges or loops. For terminology not defined here, the reader is referred to [2]. Let A be a set of vertices of a graph G. An A-*tree* is a subtree of G containing A, an A-*cut* is a set of edges meeting every A-tree in G, and A is called k-*edge-connected* in G if there is no A-cut of less than k edges. By Menger's Theorem (see [2]), a set $\{a, b\}$ is k-edge-connected if and only if there are k edge disjoint $\{a, b\}$-trees (which can be chosen as a, b-paths). A similar statement is true for $A = V(G)$: By Tutte's and Nash-Williams's base packing theorem for graphs [10, 9] it follows easily that every $2k$-edge-connected graph has k edge disjoint spanning trees. In [5], I conjectured the following generalization to A-trees.

Conjecture 1 *[5] For $k \geq 0$, every $2k$-edge-connected set A of vertices in some graph G admits a family of k edge disjoint A-trees.*

For all $k \geq 2$, there are infinitely many $(2k-1)$-edge-connected $(2k-1)$-regular graphs G, and since such a graph with more than $2k$ vertices has less than the $k \cdot (|V(G)| - 1)$ edges needed for k edge disjoint spanning trees, the bound $2k$ of Conjecture 1 is best possible in general. Lau proved that every $26k$-edge-connected set A of vertices in a graph G admits k edge disjoint A-trees [6], and a bound of $24k$ is mentioned in his thesis [7]. These are the first — and up to now the best — bounds independent from $|A|$ to the edge-connectivity of A forcing k edge disjoint A-trees. However, for *bounded* $|A|$, we can possibly do something better than even $2k$:

Conjecture 2 *For $k, \ell \geq 2$, every $\lfloor \frac{2(\ell-1)}{\ell} k + \frac{\ell-2}{\ell} \rfloor$-edge-connected set A of ℓ vertices in some graph G admits a set of k edge disjoint A-trees.*

A quite similar but slightly weaker question occurred in [3]:

Question 1 *For $k, \ell \geq 2$, every k-edge-connected set A of ℓ vertices in some graph G admits a set of $\lfloor \frac{\ell}{2(\ell-1)} k \rfloor$ edge disjoint A-trees.*

An affirmative answer to Conjecture 2 for some fixed ℓ would imply an affirmative answer to Question 1 for the same ℓ, whereas the converse implication holds only if $2k+1$ as in Conjecture 2 is not divisible by ℓ (as it is the case for even ℓ). Therefore, I prefer to work with Conjecture 2. For $\ell = 2$, the statement of Conjecture 2 is just an artificial way of rephrasing Menger's Theorem (see above). For $\ell = 3$ it has been proven [5]. In [3], it has been proven that, for $k \geq 2$ every k-edge-connected set A of ℓ vertices admits $\lfloor \alpha_\ell k \rfloor$ edge disjoint A-trees, where α_ℓ is defined recursively by $\alpha_2 = 1$ and $\alpha_{\ell+1} = \alpha_\ell - \alpha_\ell^2/4$. For $\ell \leq 5$ we obtain $\alpha_\ell > 1/2$, namely $\alpha_3 = 3/4$, $\alpha_4 = 39/64$, and $\alpha_5 = 8463/16384$, so that Conjecture 1 is true for $\ell \leq 5$ — indeed with bounds better than $2k$. The following result also supports Conjecture 2.

Theorem 1 *[5] For $k, \ell \geq 2$, every $\lfloor \frac{2(\ell-1)}{\ell} k + \frac{\ell-2}{\ell} \rfloor$-edge-connected graph on ℓ vertices has k edge disjoint spanning trees.*

That is, the statement of Conjecture 2 is true if $A = V(G)$. Moreover, the bound to the edge-connectivity is sharp for all possible pairs of ℓ, k [5].

The main result of this paper is that the statement of Conjecture 2 is true for $\ell = 4$. That is, we prove that every $\lceil \frac{3k}{2} \rceil$-edge-connected set A of four vertices of some graph G admits a set of k edge disjoint A-trees in G. This lifts the bound of $\lceil \frac{64k}{39} \rceil$ implied by $\alpha_3 = 39/64$ to the optimum. Using the recursion for α with the improved entry point $\alpha_4 = \frac{2}{3}$, one calculates $\alpha_5 = \frac{5}{9}$, so that every $\lceil \frac{9k}{5} \rceil$-edge-connected set A of five vertices in some graph G admits a set of k edge disjoint A-trees. Even with this improved numerical data, Conjecture 1 remains open for $\ell = 6$ (and all larger ℓ), as $\alpha_6 = \frac{155}{324} < 1/2$.

2 Binary bridges and terminal degrees

For a set A of vertices of some graph G, an *A-bridge* is a subgraph B of G which is formed by either an edge connecting two vertices from A or by the edges incident with the vertices of some component of $G - A$. B is called *binary* if it is a tree and the vertices in $V(B) - A$ have degree 3 in G. In this section we will see how to reduce the problem of finding k edge disjoint A-trees for a λ-edge-connected set A to the case that all A-bridges are binary and, moreover, all vertices in A have degree λ. Whereas the reduction to all A-bridges being binary has been considered earlier in [4], the regularity constraint to the vertices of A needs a new argument.

Let us collect some prerequisites for the proof. An edge e of G is called *essential (for A being λ-edge-connected in G)*, if A is λ-edge-connected in G but not in $G - e$. It is easy to see that this is equivalent to the statement that e is contained in some A-cut of cardinality λ. A is *minimally λ-edge-connected* in G if every edge of G is essential for A being λ-edge-connected in G. — The following result is from [4].

Theorem 2 *[4] Let A be a λ-edge-connected set of vertices in some graph G and let B be an A-bridge such that every vertex $x \in V(B) - A$ has degree* 3 *in G and the three edges incident with x are essential for A being λ-edge-connected in G. Then B is binary.*

Let x be a vertex of some graph G. A *splitting at x* is a pair $p = (wx, xy)$ of distinct edges incident with x. The graph $G(p)$ obtained from $G - \{wx, xy\}$ by adding a single new *bypass edge* from w to y is said to be obtained from G by *performing p*. The splitting p is called *admissible* if, for all distinct $a, b \in V(G) - \{x\}$, every $\{a, b\}$-cut in $G(p)$ is at least as large as a smallest $\{a, b\}$-cut in G. That is, an admissible splitting does not decrease the connectivity of pairs not involving x. The following theorem from [8] is a fundamental result on the existence of admissible splittings.

Theorem 3 *[8] Let x be a vertex of some graph G which is neither isolated nor incident with a cut edge. Then there exists an admissible splitting at x.*

Now we are prepared to prove the following.

Theorem 4 *Let $\ell, \lambda, k \geq 0$ be integers. Then the following statements are equivalent.*

(i) Every λ-edge-connected set A of ℓ vertices in some graph G admits a family of k edge disjoint A-trees.

(ii) Every λ-edge-connected set A of ℓ vertices of degree λ in some graph G such that every A-bridge is binary admits a family of k edge disjoint A-trees.

Proof. Statement (ii) is obviously necessary for G. Suppose that (ii) holds and assume, to the contrary, that (i) does not. Then there exist G, A such that A with $|A| = \ell$ is λ-edge-connected but G does not admit k edge disjoint A-trees, and among them we choose G, A such that $(\alpha(G, A), |E(G)|, |V(G)|)$ is lexicographically minimal, where $\alpha(G, A) := \sum_{a \in A} d_G(a)$.

We may assume that $k \geq 1$ and $|A| \geq 2$, as (i) is trivially true if $k = 0$ or $\ell < 2$. For $k \geq 1$, $\lambda \geq k$ follows, as otherwise the union of λ many binary trees, each with A being its set of end vertices, and pairwise disjoint outside A, would satisfy the premise of (ii) but not the conclusion. As there exists an A-tree if (and only if) A is 1-edge-connected, (i) holds for $k = 1$, too; we thus may assume $\lambda \geq k \geq 2$. In fact, $\lambda \geq 3$, as $\lambda = 2$ forces $k = 2$ and the cycle of length $|A|$ on A satisfies the premise of (ii) but not the conclusion.

Claim 1. G is 2-edge-connected.

For otherwise, G had a cut C on less than two edges. C cannot be an A-cut as $\lambda > 1$, so $G - C$ had a component $X \subseteq V(G) - A$; since no A-path can intersect X, A remains λ-edge-connected in $G - X$. Since $\alpha(G - X, A) \leq \alpha(G, A)$, $|E(G - X)| \leq |E(G)$, and $|V(G - X)| < |V(G)|$, $G - X$ contains k edge disjoint A-trees by choice of G, A, and they survive in G, contradicting the choice of G, A. This proves Claim 1.

Claim 2. A is minimally λ-edge-connected in G.

For otherwise there was an edge $e \in E(G)$ such that A is λ-edge-connected in $G - e$. Since $\alpha(G - e, A) \leq \alpha(G, A)$ and $|E(G - e)| < E(G)$, $G - e$ contains k edge disjoint A-trees by choice of G, A, and they survive in G, contradicting the choice of G, A. This proves Claim 2.

Claim 3. Every A-bridge is binary.

Consider $x \in V(G) - A$. Then $d_G(x) = 3$, for otherwise, by Claim 1 and Theorem 3, there exists an admissible splitting p at x; A is λ-edge-connected in $G(p)$. Since $\alpha(G(p), A) \leq \alpha(G, A)$ and $|E(G(p)| < |E(G)|$, $G(p)$ contains k edge disjoint A-trees by choice of G, A. By replacing the bypass edge with the two splitting edges in (at most one of) these trees and taking spanning trees of the results, we obtain k edge disjoint A-trees in G, contradicting the choice of G, A. — By Claim 2 and Theorem 2, Claim 3 follows.

Since we are supposing that (ii) holds, we deduce from Claim 3 that there must be a vertex b in A with $d_G(b) \geq \lambda + 1 \geq 4$. By Claim 1 and Theorem 3, there exists an admissible splitting p at b. $A - \{b\}$ is λ-edge-connected in $G(p)$.

By Claim 3, every A-bridge containing b contains exactly one edge incident with b. Hence there are exactly $d_G(b)$ distinct A-bridges containing b. Each of them contains an $A - \{b\}, b$-path, and hence there is a set $\mathcal{Q}$ of $d_G(b) - 2$ many edge disjoint $A - \{b\}, b$-paths in G which avoid the two bridges B, B' which contain the splitting edges from p.

If $d_G(b) \geq \lambda + 2$ then the paths from $\mathcal{Q}$ ensure that A is λ-edge-connected in $G(p)$. Since $\alpha(G(p), A) < \alpha(G, A)$, $G(p)$ contains k edge disjoint A-trees by choice of G, A. By replacing the bypass edge with the two splitting edges in (at most one of) these trees and taking spanning trees of the results, we obtain k edge disjoint A-trees in G, contradicting the choice of G, A.

So let $d_G(b) = \lambda + 1$. There exists a vertex $a \in (A - \{b\}) \cap V(B)$, so that we find a path R from a in $B - \{b\}$ to the neighbor w of b in B. w is an end vertex of the bypass edge, and R is edge disjoint from any path in $\mathcal{Q}$. We subdivide the bypass edge in $G(p)$ by a new vertex x and add a new edge e^+ from x to b. Let G^+ be the graph obtained this way. $A - \{b\}$ is λ-edge-connected in G^+ and the paths from $\mathcal{Q}$ are paths in G^+. We extend R to an $A - \{b\}, b$-path R^+ disjoint from $\mathcal{Q}$ by appending x, b. Now the λ paths from $\mathcal{Q} \cup \{R^+\}$ certify that A is λ-edge-connected in G^+. Since $\alpha(G^+, A) < \alpha(G, A)$, G^+ contains k edge disjoint A-trees by choice of G, A. By contracting e^+ we recover G (if we identify the subdivision edges introduced with x with the respective splitting edges), so that contracting e in (at most one of) the trees and taking spanning trees of the results yields k edge disjoint A-trees in G, contradicting the choice of G, A. □

3 *A*-bridges which are *A*-trees

In this section we show how to reduce binary A-bridges which are, at the same time, A-trees. This is an easy consequence of the following result from [4].

Theorem 5 *[4] Let A be a λ-edge-connected set of vertices in some graph G and let B be a binary A-bridge such that every edge of B is essential for A being λ-edge-connected. Then A is $(\lambda - 1)$-edge-connected in $G - E(B)$.*

This yields almost immediately the following.

Theorem 6 *Let $\ell \geq 0$ be an integer and $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(0) \geq 0$ and $f(k+1) \geq f(k)+1$. Then the following statements are equivalent.*

(i) For every $k \geq 0$, every $f(k)$-edge-connected set A of ℓ vertices in some graph G admits a family of k edge disjoint A-trees.

(ii) For every $k \geq 0$, every $f(k)$-edge-connected set A of ℓ vertices of degree $f(k)$ in some graph G such that every A-bridge is binary and not an A-tree admits a family of k edge disjoint A-trees.

Proof. It suffices to prove that (ii) is equivalent to the statement that

(iii) for every $k \geq 0$, every $f(k)$-edge-connected set A of ℓ vertices of degree $f(k)$ in some graph G such that every A-bridge is binary admits a family of k edge disjoint A-trees,

because, for any fixed k, (iii) is equivalent to (i) by Theorem 4.

(ii) is obviously necessary for (iii). Suppose that (ii) holds and assume, to the contrary, that (iii) does not. Then there exist k, G, A such that A with $|A| = \ell$ is $f(k)$-edge-connected in G, every A-bridge is binary, but there is no set of k edge disjoint A-trees. We take them such that $(k, |E(G)|)$ is lexicographically minimal. By (ii), there must be an A-bridge B which is, at the same time, an A-tree.

Assume, to the contrary, that B contains an edge e such that A is $f(k)$-edge-connected in $G - e$. Any $y \in V_G(e) - A$ has exactly two neighbors x_y, z_y in $G - e$; let G^- be obtained from $G - e$ by deleting any such y and adding a new edge e_y connecting x_y, z_y. (Alternatively, contract one edge incident with any such y.) It is easy to see that A is $f(k)$-edge-connected in G^-, and that every A-bridge is binary. Since $|E(G^-)| \leq |E(G - e)| < |E(G)|$, we find k edge disjoint A-trees in $G - e$ by choice of k, G, A. By replacing an edge e_y with the path x_y, y, z_y in (at most two of) these trees, we obtain k edge disjoint A-trees in G, a contradiction.

Hence every edge of B is essential for A being $f(k)$-edge-connected in G. By Theorem 5, A is $(f(k) - 1)$-edge-connected in $G - E(B)$, and thus $f(k - 1)$-edge-connected in $G^- := (G - E(B)) - (V(B) - A)$. Obviously, every A-bridge of G^- is binary. By choice of k, G, A, there exist $k - 1$ edge disjoint A-trees in G^-, which form, together with the A-tree B, a family of k edge disjoint A-trees in G — a contradiction. □

Theorem 1 and Theorem 6 provide an alternative way of proving the statement of Conjecture 2 for $\ell = 3$:

Theorem 7 *[5] For $k \geq 0$, every $\lfloor \frac{4}{3}k + \frac{1}{3} \rfloor$-edge-connected set $\{a, b, c\}$ of vertices in some graph G admits a family of k edge disjoint $\{a, b, c\}$-trees.*

Proof. By Theorem 1 and trivial reasons in case of $k < 2$, (ii) of Theorem 6 is true for $\ell = 3$ and $f(k) = \lfloor \frac{4}{3}k + \frac{1}{3} \rfloor$, so that (i) of Theorem 6 for these objects follows — which is the statement to be proven. □

4 Four terminals

Let us now prove the statement of Conjecture 2 for $\ell = 4$. We would like to apply Theorem 6 in the same way as we did before in the proof of Theorem 7. However, the statement of (ii) in Theorem 6 for $\ell = 4$ does not collapse to a statement on spanning trees of small graphs as it did for $\ell = 3$, but to a more difficult one given in the following theorem. Its proof heavily uses the regularity assumption to the vertices in A, which thus might be helpful to attack Conjecture 2 or Conjecture 1 in the future.

Theorem 8 *Let $k \geq 0$ and A be a $\lceil \frac{3}{2}k \rceil$-edge-connected set of four vertices in a graph G such that all vertices in A have degree $\lceil \frac{3}{2}k \rceil$ and every A-bridge is either a tree K_2 or a tree $K_{1,3}$, with all end vertices being contained in A. Then there exists a family of k edge disjoint A-trees in G.*

Proof. We prove the statement by induction on k. It is obviously true for $k \leq 1$ (but not sharp for $k = 1$). Now let $k > 1$, set $\lambda := \lceil \frac{3}{2}k \rceil \geq 3$, and let $A = \{1, 2, 3, 4\}$. Throughout, the A-bridges isomorphic to K_2 are called *small*, the others are *big*. For $i_1, \dots i_\ell \in A$, let $\mathcal{B}_{i_1 \dots i_\ell}$ denote the set of A-bridges B with $V(B) \cap A = \{i_1, \dots, i_\ell\}$, let $\mathcal{B}^-$ denote the set of all small bridges, and let $\mathcal{B}^+$ denote the set of all big bridges. We may assume that $\mathcal{B}^+ \neq \emptyset$, as otherwise Theorem 1 yields the statement. Moreover, $|\mathcal{B}^+|$ is even since $4\lambda = \sum_{a \in A} d_G(a) = 2|\mathcal{B}^-| + 3|\mathcal{B}^+|$; in particular, $|\mathcal{B}^+| \geq 2$.

Let p be a partition of A into two classes. For every A-bridge B intersecting both classes of p, let $e_{p,B}$ be the unique edge connecting a vertex from one class to either a vertex from the other class or a common neighbor outside A of two distinct vertices of the other class. Let C_p be the set of all edges $e_{p,B}$ where B is any A-bridge intersecting both classes of p. It follows easily that C_p is a minimal A-cut in G, containing exactly one edge of either A-bridge intersecting both classes of p. In particular, $|C_p| \geq \lambda$. If p is unbalanced, that is, $p = \{\{a\}, A - \{a\}\}$ for some $a \in A$, then $C_p = E_G(a)$, and we call C_p a *small cut*. Otherwise, C_p is a *big cut*. Hence there are four small cuts and three big cuts. An A-cut C is called *tight* if $|C| = \lambda$. Since $d_G(a) = \lambda$ for all $a \in A$, every small cut is tight, whereas a big cut need not to be tight.

It follows from the definitions that the unique edge of a small bridge is contained in precisely two of the three big cuts, and that each edge of a big bridge is in precisely one of them. Therefore, the sum of the sizes of the three big cuts is equal to $\sum_{a \in A} d_G(a)$, that is, equal to 4λ. Hence the average size of a big cut is $\frac{4\lambda}{3}$, which implies that there is at least one non-tight big cut.

Claim 1. If there are distinct A-bridges B^1, B^2, B^3, B^4 such that $T_1 := B^1 \cup B^2$ and $T_2 := B^3 \cup B^4$ are connected subgraphs of G containing A and such that $T_1 \cup T_2$ contains at most three edges from every small cut and from every tight big cut and at most four edges from every non-tight big cut then G has a family of k edge disjoint A-spanning trees.

Let $G^- := G - E(T_1 \cup T_2) - (V(T_1 \cup T_2) - A)$. Then the A-bridges of G^- are the A-bridges of G distinct from $B^1, \ldots, B^4$. Suppose, to the contrary, that A is not $(\lambda - 3)$-edge-connected in G^-. Then there exists a partition $\{X, Y\}$ of $V(G^-)$ such that both X, Y intersect A and $|E_G(X,Y)| \leq \lambda - 4$. Among them we choose $\{X, Y\}$ in such a way that $E_G(X,Y) =: C^-$ is as small as possible. Set $p := \{X \cap A, Y \cap A\}$. It follows that if a big A-bridge B of G^- intersects $A \cap X$ in more than one vertex then the vertex x in $V(B) - A$ is also contained in X (as otherwise $|E_G(X \cup \{x\}, Y - \{x\})| < |C^-|$); moreover, if B does not intersect $A \cap Y$ then C^- does not contain edges of B, and if B does intersect $A \cap Y$ then $e_{p,B}$ is the unique edge of B in C^-. As the same holds with the roles of X, Y being swapped, we see that C^- is the set of all edges $e_{p,B}$ such that B intersects both classes of p. It follows that $C^- \subseteq C_p$ and that $C_p = C^- \cup \{e_{p,B^i} : i \in \{1,2,3,4\},\ B^i \text{ intersects both classes of } p\}$. As $|C^-| \leq \lambda - 4$ and $|C_p| \geq \lambda$ we see that $|C_p| = \lambda$ and that C_p contains four edges from $B^1 \cup B^2 \cup B^3 \cup B^4 = T_1 \cup T_2$. But then, according to the assumption on the B^i, C_p is a non-tight big cut, contradicting $|C_p| = \lambda$.

Since $\lambda - 3 = \lceil \frac{3k}{2} - 3 \rceil = \lceil \frac{3(k-2)}{2} \rceil$, it follows by induction that G^- admits $k-2$ edge disjoint A-trees, and they form together with a spanning tree of each of T_1, T_2 the desired set of k edge disjoint A-trees in G. This proves Claim 1.

Claim 2. Every big cut has size less than 2λ.

By symmetry, it suffices to consider the big cut C_p where $p = \{\{1,2\},\{3,4\}\}$. Suppose, to the contrary, that $|C_p| \geq 2\lambda$. Elementary counting yields $|C_p| = |\mathcal{B}_{13}| + |\mathcal{B}_{14}| + |\mathcal{B}_{23}| + |\mathcal{B}_{24}| + |\mathcal{B}_{123}| + |\mathcal{B}_{124}| + |\mathcal{B}_{134}| + |\mathcal{B}_{234}| = (|E_G(1)| - |\mathcal{B}_{12}| + |E_G(2)| - |\mathcal{B}_{12}|) - |\mathcal{B}_{123}| - |\mathcal{B}_{124}| = 2\lambda - 2|\mathcal{B}_{12}| - |\mathcal{B}_{123}| - |\mathcal{B}_{124}|$, so that $\mathcal{B}_{123} = \mathcal{B}_{124} = \emptyset$. By symmetry (swap the roles of $\{1,2\},\{3,4\}$), $\mathcal{B}_{134} = \mathcal{B}_{234} = \emptyset$ — implying $\mathcal{B}^+ = \emptyset$, a contradiction. This proves Claim 2.

Claim 3. For distinct a, b from A, there exists an A-bridge containing a and b.

Suppose, to the contrary, that there is no A-bridge containing both a and b. Then there are 2λ distinct A-bridges incident with either a or b, and each of them has an edge in C_p, where $p = \{\{a,b\}, A - \{a,b\}\}$. Therefore, $|C_p| = 2\lambda$, contradicting Claim 2. This proves Claim 3.

Since the sum of the sizes of all three big cuts is 4λ, we know by Claim 2 that there is a big cut C_p such that the two big cuts distinct from C_p are non-tight. By symmetry, we may assume that $p = \{\{1,2\},\{3,4\}\}$.

Claim 4. If $|\mathcal{B}_{i_1 i_2 i_3}| > 1$ for distinct i_1, i_2, i_3 from A then G contains k edge disjoint A-trees.

By symmetry, it suffices to prove this for $i_1 = 1, i_2 = 2, i_3 = 3$. So let there be distinct big bridges $B^1, B^2 \in \mathcal{B}_{123}$.

Case 1. C_p is tight.

Since two of the bridges containing 3 but not 4 do intersect C_p, at least two of the λ many bridges containing 4 do not intersect C_p, implying that there are two distinct small bridges $B^3, B^4 \in \mathcal{B}_{34}$. Since there exist bridges containing 3 but not 4, there exists a bridge B containing 4 but not 3, too. If B is small then $B \in \mathcal{B}_{14} \cup \mathcal{B}_{24}$, and B^1, B^3, B^2, B meet the conditions of Claim 1 (for B^1, B^2, B^3, B^4). Otherwise, B is large and $B \in \mathcal{B}_{124}$, so that B^1, B^3, B, B^4 meet the conditions of Claim 1.

Case 2. C_p is not tight.

Observe that now the situation is symmetric in the big cuts, so that we can swap the roles of $1, 2, 3$ in any way we want. Suppose first that there exists a small A-bridge B containing 4. By symmetry we may assume that $B \in \mathcal{B}_{14}$. Since there exist bridges containing 1 but not 4, there exists a bridge B^3 which contains 4 but not 1, and so B^1, B, B^2, B^3 meet the conditions of Claim 1. Therefore, we may assume that every A-bridge containing 4 is big. For every $a \in \{1, 2, 3\}$, there exists a bridge containing a but not 4, and hence there exists a big bridge B^4 containing 4 but not a. It is then easy to see that B^2, B^3, B^4, B^5 (in any order) meet the conditions of Claim 1.

Hence, in any case, Claim 1 implies Claim 4.

Now suppose that C_p is tight. There exist two distinct bridges B^1, B^2, and, by Claim 4, they intersect in exactly two vertices from A. These two might be in the same class of p or not. If they are in the same class of p then we may assume by symmetry that $B^1 \in \mathcal{B}_{123}$ and $B^2 \in \mathcal{B}_{124}$. There are at least $\lambda - 1$ A-bridges distinct from B^1, B^2 containing 3 or 4, and not all of them intersect C_p; hence there exists a bridge $B^3 \in \mathcal{B}_{34}$. There is a bridge B^4 distinct from B^1, B^2 which intersects C_p. If B^4 contains 3, then B^1, B^3, B^2, B^4 meet the conditions of Claim 1; otherwise, B^4 contains 4, and B^1, B^4, B^2, B^3 meet the conditions of Claim 1. — If the two common vertices of B^1, B^2 are not in the same class of p then we may assume by symmetry that $B^1 \in \mathcal{B}_{123}$ and $B^2 \in \mathcal{B}_{134}$. Since C_p is tight, there exists a bridge B^3 containing 2 which does not intersect C_p. Symmetrically, there exists a bridge B^4 containing 4 which does not intersect C_p. It follows that $B^3 \in \mathcal{B}_{12}$ and $B^4 \in \mathcal{B}_{34}$, and hence B^1, B^4, B^2, B^3 meet the conditions of Claim 1. Hence, in the case that C_p is tight, Claim 1 provides the desired family of A-trees.

We therefore may assume that C_p is not tight, and the situation is symmetric in $1, 2, 3, 4$ again. As we have seen, $|\mathcal{B}^+|$ is an even positive number. If $|\mathcal{B}^+| \geq 4$ then, by Claim 4, $|\mathcal{B}^+| = 4$, for each $a \in A$ there exists a big bridge B^a avoiding

a, and B^1, B^2, B^3, B^4 (in any order) meet the condition of Claim 1. If $|\mathcal{B}^+| = 2$ then there are exactly two big bridges B^1, B^2, and, by Claim 4, they have exactly two vertices in common. By symmetry we may assume that $B^1 \in \mathcal{B}_{123}$ and $B^2 \in \mathcal{B}_{124}$. It is then easy to see that there are distinct small bridges B^3, B^4 such that B^3 contains 3, B^4 contains 4, and their intersection contains neither 1 nor 2. Therefore, B^1, B^4, B^2, B^3 meet the conditions of Claim 1, which finally proves the Theorem. □

The bound $\lambda := \lceil \frac{3k}{2} \rceil$ in Theorem 8 cannot be improved for $k > 1$, as there exists a $(\lambda - 1)$-regular $(\lambda - 1)$-edge-connected graph $G_{\lambda-1}$: For even $\lambda - 1$, replace every edge in a cycle of length 4 by $(\lambda - 1)/2$ edges connecting the same vertices, and for odd $\lambda - 1$, add two disjoint edges to $G_{\lambda-2}$. $G_{\lambda-1}$ has $2\lambda - 2$ edges, and, since $\lambda - 1 \leq \frac{3k-1}{2}$, a family of edge disjoint spanning trees of $G_{\lambda-1}$ has size at most $(2\lambda - 2)/3 \leq k - 1$. These examples also show that the bound of the following statement is sharp for $k > 1$.

Theorem 9 *For $k \geq 0$, every $\lceil \frac{3}{2}k \rceil$-edge-connected set $\{a, b, c, d\}$ of vertices in some graph G admits a family of k edge disjoint $\{a, b, c, d\}$-trees.*

Proof. By Theorem 8, (ii) of Theorem 6 is true for $\ell = 4$ and $f(k) = \lceil \frac{3}{2}k \rceil$, so that (i) of Theorem 6 for these objects follows — which is the statement to be proven. □

As we have observed earlier, this also yields an affirmative answer to Question 1 for $\ell = 4$. It is possible to transform the entire proof of Theorem 9 into a polytime approximation algorithm with factor 1.5 for the Steiner *tree packing problem on four terminals*, that is, given a graph and four of its vertices a, b, c, d, find a largest set of edge disjoint $\{a, b, c, d\}$-trees. It is unlikely that there is an approximation algorithm for this problem with a factor arbitrarily close to 1, as the problem is not in APX if $P \neq NP$ [1].

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