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Two-fold Mellin-Barnes transforms of Usyukina-Davydychev functions

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Abstract

In our previous paper (Nucl. Phys. B **870** (2013) 243) we showed that multi-fold Mellin-Barnes (MB) transforms of the Usyukina-Davydychev (UD) functions may be reduced to two-fold MB transforms. The MB transforms were written there as polynomials of logarithms of ratios of squares of the external momenta with certain coefficients. We also showed that these coefficients have a combinatoric origin. In this paper we present an explicit formula for these coefficients. The procedure of recovering the coefficients is based on taking the double uni-form limit in certain series of smooth functions of two variables which is constructed according to a pre-determined iterative way. The result is obtained by using basic methods of mathematical analysis. We observe that the finiteness of the limit of this iterative chain of smooth functions should reflect itself in other mathematical constructions, too, since it is not related in any way to the explicit form of the MB transforms.

Keywords: Bethe-Salpeter equation; Mellin-Barnes transform; Usyukina-Davydychev functions

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1 Introduction

Mellin-Barnes transformation is an efficient method for calculation of Feynman diagrams [1, 2, 3]. This method played important role in multi-loop calculations for the maximally supersymmetric Yang-Mills theory [4, 5] in which the class of contributing master integrals for the Feynman diagrams is reduced to the integrals corresponding to scalar ladder diagrams [4] at least at the level of the first three loops in momentum space.

The scalar ladder diagrams at any order in loops in $d = 4$ space-time dimensions have been studied for the first time in Refs. [6, 7, 8] in momentum space. The result of calculation of the momentum integrals are UD functions [7, 8]. Their MB transforms have been investigated in Refs. [9, 10]. These functions have remarkable properties, in particular they possess invariance with respect to Fourier transformation [11, 12]. Later, this property has been generalized to any three point Green's function in the massless theory via MB transform for an arbitrary spacetime dimension [9, 13]. Due to this invariance with respect to Fourier transformation the UD functions appear in the results of calculation of the Green functions in position space [14, 15, 16, 17, 18, 19, 20].

In Ref.[10] the multi-fold MB transforms of UD functions have been reduced to the two-fold MB transforms. This result allows us to simplify the analysis of the recursive property of the MB transforms of the UD functions to the analysis of the recursive property for the smooth functions that appear in the integrand of the MB transformation [10]. The MB transform of the UD function with number n turns out to be a linear combination of three MB transforms of the UD function with number $n-1$ where each of these three MB transforms depends on two independent variables $\varepsilon_1, \varepsilon_2$ in its proper well-defined manner. The coefficients in front of these combinations of the MB transforms with lower indices are singular in these two independent variables in the limit in which these variables vanish. However, these singularities cancel each other and the double uniform limit always exists and is finite for each number n.

This limit is a sum of powers of logarithms of certain arguments multiplied by derivatives of the Euler Γ-function constructed in a such way that the sum of the power of logarithm and of the derivative order is a fixed number which depends on the number of the corresponding UD function which in turn coincides with number of the rungs in the given ladder diagram. We have constructed the recursive procedure but have not considered the MB transforms of the higher UD functions in Ref. [10]. In the present paper we find the explicit form of the MB transforms of the higher UD functions by establishing the coefficients in front of the powers of logarithms. The arguments of logarithms are ratios of squares of the external incoming momenta of the ladder diagrams. As we have mentioned in Ref. [10] the coefficients have an origin in the combinatorics and are certain combinations of the combinatoric numbers C_m^n.

2 Recursive relations for MB transforms

In Ref. [10] the MB transform of the second UD function has been found in terms of the double MB transform,

$$\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) =$$

$$\frac{J}{2} \oint_C du\ dv\ x^u\ y^v \left[\frac{a(\varepsilon_1)}{\varepsilon_2 \varepsilon_3} y^{\varepsilon_2} \left[x^{\varepsilon_1} \Gamma(-u-\varepsilon_1)\Gamma(-v+\varepsilon_1) + y^{\varepsilon_1} \Gamma(-u+\varepsilon_1)\Gamma(-v-\varepsilon_1) \right] \right.$$

$$+\frac{a(\varepsilon_3)}{\varepsilon_1\varepsilon_2}\left[x^{-\varepsilon_3}\Gamma(-u+\varepsilon_3)\Gamma(-v-\varepsilon_3)+y^{-\varepsilon_3}\Gamma(-u-\varepsilon_3)\Gamma(-v+\varepsilon_3)\right]$$
$$+\frac{a(\varepsilon_2)}{\varepsilon_1\varepsilon_3}x^{\varepsilon_1}\left[x^{\varepsilon_2}\Gamma(-u-\varepsilon_2)\Gamma(-v+\varepsilon_2)+y^{\varepsilon_2}\Gamma(-u+\varepsilon_2)\Gamma(-v-\varepsilon_2)\right]\Bigg]\times$$
$$\times\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v), \qquad (1)$$

where the definition

$$a(\varepsilon)=\left[\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\right]^{-1}, \quad a_0^{(n)}=(a(\varepsilon))_{\varepsilon=0}^{(n)}$$

has been introduced. In the limit of vanishing ε_i which always are subject to the condition

$$\varepsilon_1+\varepsilon_2+\varepsilon_3=0,$$

we may write

$$\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=$$
$$\oint_C du\ dv\ x^u\ y^v\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)\left[\frac{3}{2}\left(a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)_0^{(2)}+\right.$$
$$\left.\frac{3}{2}\ln\frac{x}{y}\left(a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)_0'+\frac{1}{4}\ln^2\frac{x}{y}\Gamma(-u)\Gamma(-v)\right]. \qquad (2)$$

As we can see, a finite limit exists. This is expectable, since this expression has been constructed from another expression for which a finite limit exists.

To be more concise, we introduce another notation

$$M_1^{(u,v)}(\varepsilon)\equiv\frac{1}{2}\left[x^{\varepsilon}a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)+y^{\varepsilon}a(-\varepsilon)\Gamma(-u+\varepsilon)\Gamma(-v-\varepsilon)\right]$$
$$\times\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v).$$

With this notation, we write instead of the previous integral relation the following relation

$$\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=$$
$$J\oint_C du\ dv\ x^u\ y^v\left[\frac{1}{\varepsilon_2\varepsilon_3}y^{\varepsilon_2}M_1^{(u,v)}(\varepsilon_1)+\frac{1}{\varepsilon_1\varepsilon_2}M_1^{(u,v)}(-\varepsilon_3)+\frac{1}{\varepsilon_1\varepsilon_3}x^{\varepsilon_1}M_1^{(u,v)}(\varepsilon_2)\right].$$

The formula of Ref.[10], relating the MB transformations of the third and the second UD functions is

$$\oint_C du\ dv\ x^u\ y^v M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=$$
$$\oint_C du\ dv\ x^u\ y^v\left[\frac{1}{\varepsilon_1\varepsilon_2}x^{-\varepsilon_1}y^{-\varepsilon_2}M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\frac{J}{\varepsilon_2\varepsilon_3}x^{-\varepsilon_1}M_2^{(u,v)}(\varepsilon_1)+\frac{J}{\varepsilon_1\varepsilon_3}y^{-\varepsilon_2}M_2^{(u,v)}(\varepsilon_2)\right], \qquad (3)$$

in which we use the definitions of Ref.[10]. The finite limit of vanishing ε_i exists and is given by

$$\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0}\oint_C du\ dv\ x^u\ y^v M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=$$

$$\oint_C du\ dv\ x^u\ y^v \Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)\left[\frac{5}{12}\left(a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)_0^{(4)}+\right.$$
$$\frac{5}{6}\ln\frac{x}{y}\left(a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)_0^{(3)}+\frac{1}{2}\ln^2\frac{x}{y}\left(a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)_0^{(2)}$$
$$\left.+\frac{1}{12}\ln^3\frac{x}{y}\left(a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)_0'\right]. \tag{4}$$

This limit should be taken after substituting the expression for M_2 of Eq.(1) into the expression for M_3 of Eq. (3). The coefficient J is defined in Ref.[10] as

$$J=\frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)}. \tag{5}$$

According to formulae of Section 4.4 of Ref.[10], we have the expression for the MB transform M_4 of the fourth UD function in terms of the MB transform M_3 of the third UD function as

$$\oint_C du\ dv\ x^u\ y^v M_4^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\oint_C du\ dv\ x^u\ y^v\frac{J}{\varepsilon_2\varepsilon_3}M_3^{(u,v)}(\varepsilon_1)$$
$$+\oint_C du\ dv\ x^u\ y^v\frac{1}{\varepsilon_1\varepsilon_2}M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\oint_C du\ dv\ x^u\ y^v\frac{J}{\varepsilon_1\varepsilon_3}M_3^{(u,v)}(\varepsilon_2)=$$
$$\oint_C du\ dv\ x^u\ y^v\left[\frac{J}{\varepsilon_2\varepsilon_3}M_3^{(u,v)}(\varepsilon_1)+\frac{1}{\varepsilon_1\varepsilon_2}M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\frac{J}{\varepsilon_1\varepsilon_3}M_3^{(u,v)}(\varepsilon_2)\right]. \tag{6}$$

By going further according to the construction described in Ref.[10] for the higher UD functions with number $n>4$ we may write

$$\oint_C du\ dv\ x^u\ y^v M_n^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\oint_C du\ dv\ x^u\ y^v\frac{J}{\varepsilon_2\varepsilon_3}M_{n-1}^{(u,v)}(\varepsilon_1)$$
$$+\oint_C du\ dv\ x^u\ y^v\frac{1}{\varepsilon_1\varepsilon_2}M_{n-1}^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\oint_C du\ dv\ x^u\ y^v\frac{J}{\varepsilon_1\varepsilon_3}M_{n-1}^{(u,v)}(\varepsilon_2)$$
$$=\oint_C du\ dv\ x^u\ y^v\left[\frac{J}{\varepsilon_2\varepsilon_3}M_{n-1}^{(u,v)}(\varepsilon_1)+\frac{1}{\varepsilon_1\varepsilon_2}M_{n-1}^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\frac{J}{\varepsilon_1\varepsilon_3}M_{n-1}^{(u,v)}(\varepsilon_2)\right]. \tag{7}$$

To simplify the presentation in the previous ladder construction for higher UD functions we define

$$f(\varepsilon)=\frac{1}{2}a(\varepsilon)\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v). \tag{8}$$

Since the contour of integration in the MB transformations of UD functions in Eqs.(1),(3) and (7) passes between the leftmost of the right poles and the rightmost of the left poles in the planes of complex variables u and v which are variables of integration, we may work with the limits in Eqs. (2) and (4) at the level of integrands. The dependence on the integration variables u and v may be omitted to simplify the analysis since this dependence on u and v follows from the dependence of M_1 on them for the higher number of n in M_n. Due to this observation it is convenient to establish a new notation

$$M_n^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)\equiv\Delta_n(\varepsilon_1,\varepsilon_2,\varepsilon_3).$$

We may analyse functions $\Delta_n(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ as certain functions of three variables and represent the ladder relations of Eqs.(6) and (7) in the form

$$\begin{aligned}
\Delta_1(\varepsilon) &= x^{\varepsilon} f(\varepsilon) + y^{\varepsilon} f(-\varepsilon) \\
\Delta_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= J\left[\frac{1}{\varepsilon_2\varepsilon_3} y^{\varepsilon_2}\Delta_1(\varepsilon_1) + \frac{1}{\varepsilon_1\varepsilon_2}\Delta_1(-\varepsilon_3) + \frac{1}{\varepsilon_1\varepsilon_3} x^{\varepsilon_1}\Delta_1(\varepsilon_2)\right] \\
\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \frac{1}{\varepsilon_1\varepsilon_2} y^{-\varepsilon_2} x^{-\varepsilon_1}\Delta_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{J}{\varepsilon_2\varepsilon_3} x^{-\varepsilon_1}\Delta_2(\varepsilon_1) + \frac{J}{\varepsilon_1\varepsilon_3} y^{-\varepsilon_2}\Delta_2(\varepsilon_2) \\
\Delta_4(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \frac{J}{\varepsilon_2\varepsilon_3}\Delta_3(\varepsilon_1) + \frac{1}{\varepsilon_1\varepsilon_2}\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{J}{\varepsilon_1\varepsilon_3}\Delta_3(\varepsilon_2). \qquad (9)
\end{aligned}$$

For an arbitrary number $n > 4$ we may write

$$\Delta_n(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{J}{\varepsilon_2\varepsilon_3}\Delta_{n-1}(\varepsilon_1) + \frac{1}{\varepsilon_1\varepsilon_2}\Delta_{n-1}(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{J}{\varepsilon_1\varepsilon_3}\Delta_{n-1}(\varepsilon_2).$$

In the next section we calculate the values $\Delta_n(0)$ of the finite uniform double limit

$$\Delta_n(0) = \lim_{\varepsilon_1\to 0,\varepsilon_2\to 0}\Delta_n(\varepsilon_1,\varepsilon_2,\varepsilon_3).$$

These values will correspond to the representation of the MB transformations of UD functions described in Conclusion of Ref.[10].

3 $\Delta_n(0)$ in terms of differential operator

According to Eq. (9), the expression for Δ_2 may explicitly be written as

$$\begin{aligned}
J^{-1}\Delta_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{1}{\varepsilon_2\varepsilon_3} y^{\varepsilon_2}\left[x^{\varepsilon_1} f(\varepsilon_1) + y^{\varepsilon_1} f(-\varepsilon_1)\right] + \frac{1}{\varepsilon_1\varepsilon_2}\left[x^{-\varepsilon_3} f(-\varepsilon_3) + y^{-\varepsilon_3} f(\varepsilon_3)\right] \\
+\frac{1}{\varepsilon_1\varepsilon_3} x^{\varepsilon_1}\left[x^{\varepsilon_2} f(\varepsilon_2) + y^{\varepsilon_2} f(-\varepsilon_2)\right].
\end{aligned}$$

To simplify the analysis, we introduce a notation $\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) \equiv y^{-\varepsilon_2} x^{-\varepsilon_1}\Delta_2(\varepsilon_1,\varepsilon_2,\varepsilon_3)$. For this quantity we may write

$$\begin{aligned}
J^{-1}\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{1}{\varepsilon_2\varepsilon_3}\left[f(\varepsilon_1) + \omega^{-\varepsilon_1} f(-\varepsilon_1)\right] + \frac{1}{\varepsilon_1\varepsilon_2}\left[\omega^{\varepsilon_2} f(\varepsilon_1+\varepsilon_2) + \omega^{-\varepsilon_1} f(-\varepsilon_1-\varepsilon_2)\right] \\
+\frac{1}{\varepsilon_1\varepsilon_3}\left[\omega^{\varepsilon_2} f(\varepsilon_2) + f(-\varepsilon_2)\right],
\end{aligned}$$

where we have defined $\omega \equiv x/y$. For the future use, it is more convenient to represent $\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ in the form

$$\begin{aligned}
J^{-1}\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{\omega^{-\varepsilon_1}}{\varepsilon_1}\frac{\omega^{\varepsilon_1+\varepsilon_2} f(\varepsilon_1+\varepsilon_2) - \omega^{\varepsilon_1} f(\varepsilon_1)}{\varepsilon_2} + \frac{\omega^{-\varepsilon_1}}{\varepsilon_1}\frac{f(-\varepsilon_1-\varepsilon_2) - f(-\varepsilon_1)}{\varepsilon_2} \\
+\frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\left(\omega^{-\varepsilon_1} f(-\varepsilon_1) + f(\varepsilon_1) - \omega^{\varepsilon_2} f(\varepsilon_2) - f(-\varepsilon_2)\right).
\end{aligned}$$

We can take the limit with respect to the second variables ε_2 and the result is

$$\tilde{\Delta}_2(\varepsilon_1) = \lim_{\varepsilon_2\to 0}\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{1}{\varepsilon_1}\omega^{-\varepsilon_1}\left(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1)\right)' + \frac{1}{\varepsilon_1^2}\left(f(\varepsilon_1)+\omega^{-\varepsilon_1}f(-\varepsilon_1)-2f(0)\right). \tag{10}$$

The quantity

$$\Delta_2(0) = \tilde{\Delta}_2(0) = \lim_{\varepsilon_2\to 0,\varepsilon_1\to 0}\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{1}{2}\ln^2\omega f(0) + 3\ln\omega f^{(1)}(0) + 3f^{(2)}(0) \tag{11}$$

may be written in another form

$$\begin{aligned}\Delta_2(0) &= \frac{1}{2}\left[2f^{(2)}(0)+2\ln\omega f^{(1)}(0)+\ln^2\omega f(0)\right] + 2f^{(2)}(0)+2\ln\omega f^{(1)}(0)\\ &= \frac{1}{2}\left[f^{(2)}(\varepsilon)+\omega^{-\varepsilon}\ln^2\omega f(-\varepsilon)+2\omega^{-\varepsilon}\ln\omega f^{(1)}(-\varepsilon)+\omega^{-\varepsilon}f^{(2)}(-\varepsilon)\right]_{\varepsilon=0}\\ &\quad+\left[\ln\omega f^{(1)}(\varepsilon)+f^{(2)}(\varepsilon)+\omega^{-\varepsilon}\ln\omega f^{(1)}(-\varepsilon)+\omega^{-\varepsilon}f^{(2)}(-\varepsilon)\right]_{\varepsilon=0}\\ &= \frac{1}{2}\left[\partial_\varepsilon^2\omega^{-\varepsilon}(\omega^\varepsilon f(\varepsilon)+f(-\varepsilon))\right]_{\varepsilon=0} + \left[\partial_\varepsilon\omega^{-\varepsilon}\partial_\varepsilon(\omega^\varepsilon f(\varepsilon)+f(-\varepsilon))\right]_{\varepsilon=0}\\ &= \frac{1}{2}\left[\partial_\varepsilon[\partial_\varepsilon\omega^{-\varepsilon}+2\omega^{-\varepsilon}\partial_\varepsilon](\omega^\varepsilon f(\varepsilon)+f(-\varepsilon))\right]_{\varepsilon=0}.\end{aligned} \tag{12}$$

Neither this formula nor Eq. (11) play any important role in the further construction. However, representation (12) is necessary to observe that it is a particular case of the general formula for an arbitrary number n of $\Delta_n(0)$.

The next step is to do the following operation

$$\begin{aligned}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}J^{-1}\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \frac{1}{2\varepsilon_1}\omega^{-\varepsilon_1}\partial_{\varepsilon_1}^2(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))\\ -\frac{1}{\varepsilon_1^3}(\omega^{-\varepsilon_1}f(-\varepsilon_1)+f(\varepsilon_1)-2f(0)) &- \frac{1}{\varepsilon_1^2}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2)).\end{aligned} \tag{13}$$

As may be seen from Eq. (11), the value $\tilde{\Delta}_2[\varepsilon_1,\varepsilon_2,\varepsilon_3]$ does not have any singularity in variables ε_1 and ε_2. The same statement is true for its derivative with respect to the variable ε_2.

The operation (13) is necessary to find a value of the following term in the chain of functions $\Delta_n(0)$. Indeed, for Δ_3 we may write

$$\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{1}{\varepsilon_1\varepsilon_2}\left(\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3)-J\tilde{\Delta}_2(\varepsilon_1)\right) + \frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\left(J\tilde{\Delta}_2(\varepsilon_1)-J\tilde{\Delta}_2(\varepsilon_2)\right).$$

Taking into account Eq.(13), we may write

$$\begin{aligned}\Delta_3(\varepsilon_1) = \lim_{\varepsilon_2\to 0}J^{-1}\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \frac{1}{2\varepsilon_1^2}\omega^{-\varepsilon_1}\partial_{\varepsilon_1}^2(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))\\ -\frac{1}{\varepsilon_1^4}(\omega^{-\varepsilon_1}f(-\varepsilon_1)+f(\varepsilon_1)-2f(0)) &- \frac{1}{\varepsilon_1^3}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))\\ &+\frac{1}{\varepsilon_1^2}\left(\tilde{\Delta}_2(\varepsilon_1)-\tilde{\Delta}_2(0)\right).\end{aligned} \tag{14}$$

This quantity by construction does not have any singularity in the variable ε_1. The constants like $\tilde{\Delta}_2(0)$ and $\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$ are multiplied by the negative powers of ε_1 and should disappear in Eq. (14) at the end. From Eq. (10) we conclude that the second term in Eq.(14) will be canceled by the corresponding term in $\tilde{\Delta}_2(\varepsilon_1)$ and we conclude from Eq. (14) that

$$\Delta_3(0)=\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0}J^{-1}\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\lim_{\varepsilon_1\to 0}\left[\frac{1}{2\varepsilon_1^2}\omega^{-\varepsilon_1}\partial^2_{\varepsilon_1}(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))\right.$$
$$\left.+\frac{1}{\varepsilon_1^3}\omega^{-\varepsilon_1}\partial_{\varepsilon_1}(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))-\frac{1}{\varepsilon_1^3}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))-\frac{1}{\varepsilon_1^2}\tilde{\Delta}_2(0)\right]=$$
$$\lim_{\varepsilon\to 0}\frac{1}{4!}\partial^2_\varepsilon[4\partial_\varepsilon\omega^{-\varepsilon}\partial_\varepsilon+6\omega^{-\varepsilon}\partial^2_\varepsilon](\omega^\varepsilon f(\varepsilon)+f(-\varepsilon)).$$

As we can see, the result has a similar structure to Eq. (12), that is a differential operator of certain structure acting on $\omega^\varepsilon f(\varepsilon)+f(-\varepsilon)$. We will show that such a structure survives in more complicated cases.

Repeating for quantity $\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ steps which we did for $\tilde{\Delta}_2(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ we may write

$$\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}J^{-1}\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\frac{1}{6\varepsilon_1^2}\omega^{-\varepsilon_1}\partial^3_{\varepsilon_1}(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))$$
$$+\frac{1}{\varepsilon_1^5}(\omega^{-\varepsilon_1}f(-\varepsilon_1)+f(\varepsilon_1)-2f(0))+\frac{1}{\varepsilon_1^4}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$$
$$-\frac{1}{2\varepsilon_1^3}\lim_{\varepsilon_2\to 0}\partial^2_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$$
$$-\frac{1}{\varepsilon_1^3}\left(\tilde{\Delta}_2(\varepsilon_1)-\tilde{\Delta}_2(0)\right)-\frac{1}{\varepsilon_1^2}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}\tilde{\Delta}_2(\varepsilon_2). \tag{15}$$

Then, for Δ_4 we obtain

$$\Delta_4(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\frac{1}{\varepsilon_1\varepsilon_2}\left(\Delta_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)-J\Delta_3(\varepsilon_1)\right)+\frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\left(J\Delta_3(\varepsilon_1)-J\Delta_3(\varepsilon_2)\right).$$

Taking into account Eq. (15) we may write an analog of Eq.(14) for Δ_3 as

$$\Delta_4(\varepsilon_1)=\lim_{\varepsilon_2\to 0}J^{-1}\Delta_4(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\frac{1}{6\varepsilon_1^3}\omega^{-\varepsilon_1}\partial^3_{\varepsilon_1}(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))$$
$$+\frac{1}{\varepsilon_1^6}(\omega^{-\varepsilon_1}f(-\varepsilon_1)+f(\varepsilon_1)-2f(0))+\frac{1}{\varepsilon_1^5}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$$
$$-\frac{1}{2\varepsilon_1^4}\lim_{\varepsilon_2\to 0}\partial^2_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$$
$$-\frac{1}{\varepsilon_1^4}\left(\tilde{\Delta}_2(\varepsilon_1)-\tilde{\Delta}_2(0)\right)-\frac{1}{\varepsilon_1^3}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}\tilde{\Delta}_2(\varepsilon_2)+\frac{1}{\varepsilon_1^2}\left(\Delta_3(\varepsilon_1)-\Delta_3(0)\right). \tag{16}$$

This quantity by construction does not have any singularity in the variable ε_1. The constants like $\tilde{\Delta}_2(0)$, $\Delta_3(0)$, $\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$ and $\lim_{\varepsilon_2\to 0}\partial^2_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$ are multiplied by the negative powers of ε_1 and should finally disappear in Eq. (16). From Eq. (14) we conclude that the second term on the r.h.s. of Eq. (16) will be canceled by the corresponding term in $\Delta_3(\varepsilon_1)$,

while the first term in the last line of Eq. (16) will be canceled by another term in $\Delta_3(\varepsilon_1)$. We conclude from Eq. (16) that

$$\Delta_4(0) = \lim_{\varepsilon_2\to 0,\varepsilon_1\to 0} J^{-1}\Delta_4(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \lim_{\varepsilon_1\to 0}\left[\frac{1}{6\varepsilon_1^3}\omega^{-\varepsilon_1}\partial^3_{\varepsilon_1}(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1))\right.$$
$$+\frac{1}{2\varepsilon_1^4}\omega^{-\varepsilon_1}\partial^2_{\varepsilon_1}(\omega^{\varepsilon_1}f(\varepsilon_1)+f(-\varepsilon_1)) - \frac{1}{2\varepsilon_1^4}\lim_{\varepsilon_2\to 0}\partial^2_{\varepsilon_2}(\omega^{\varepsilon_2}f(\varepsilon_2)+f(-\varepsilon_2))$$
$$\left.-\frac{1}{\varepsilon_1^3}\lim_{\varepsilon_2\to 0}\partial_{\varepsilon_2}\tilde{\Delta}_2(\varepsilon_2) - \frac{1}{\varepsilon_1^2}\Delta_3(0)\right] =$$
$$= \lim_{\varepsilon\to 0}\frac{1}{6!}\partial^3_\varepsilon(15\partial_\varepsilon\omega^{-\varepsilon}\partial^2_\varepsilon + 20\omega^{-\varepsilon}\partial^3_\varepsilon)(\omega^\varepsilon f(\varepsilon)+f(-\varepsilon)).$$

We may proceed further for the higher number n and find the following relations

$$\begin{aligned}
\tilde{\Delta}_2(0) &= \lim_{\varepsilon\to 0}\frac{1}{2!}\partial_\varepsilon\left[C_2^0\partial_\varepsilon\omega^{-\varepsilon}\partial^0_\varepsilon + C_2^1\omega^{-\varepsilon}\partial_\varepsilon\right](\omega^\varepsilon f(\varepsilon)+f(-\varepsilon))\\
\Delta_3(0) &= \lim_{\varepsilon\to 0}\frac{1}{4!}\partial^2_\varepsilon\left[C_4^1\partial_\varepsilon\omega^{-\varepsilon}\partial_\varepsilon + C_4^2\omega^{-\varepsilon}\partial^2_\varepsilon\right](\omega^\varepsilon f(\varepsilon)+f(-\varepsilon))\\
\Delta_4(0) &= \lim_{\varepsilon\to 0}\frac{1}{6!}\partial^3_\varepsilon\left[C_6^2\partial_\varepsilon\omega^{-\varepsilon}\partial^2_\varepsilon + C_6^3\omega^{-\varepsilon}\partial^3_\varepsilon\right](\omega^\varepsilon f(\varepsilon)+f(-\varepsilon)).
\end{aligned}$$

The result for an arbitrary $n > 4$ is

$$\Delta_n(0) = \lim_{\varepsilon\to 0}\frac{1}{(2(n-1))!}\partial^{n-1}_\varepsilon\left[C^{n-2}_{2(n-1)}\partial_\varepsilon\omega^{-\varepsilon}\partial^{n-2}_\varepsilon + C^{n-1}_{2(n-1)}\omega^{-\varepsilon}\partial^{n-1}_\varepsilon\right](\omega^\varepsilon f(\varepsilon)+f(-\varepsilon)).$$

4 Conclusion

The explicit form of the coefficients found in the present paper allows us on one hand, to write a sum of all the ladder diagrams which presents the solution to the Bethe-Salpeter equation in case if f is chosen to be Euler Γ function as in Ref. [21], and on the other hand, to write the explicit form of the integration formulae derived in Ref. [10]. All the values $\Delta_n(0)$ have been found in the form of the differential operators acting on the first term of the chain of MB transforms. It is plausible that for other functions f distinct from Γ function, these recursive relations may be mapped to recursive relations of other integrable systems in quantum mechanics or condensed matter theory. As the consequence of the main result of the paper, we conclude that the higher UD functions can be obtained via application of certain differential operator to a simple generalization of the first UD function.

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