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Dedicated to Lothar Reichel on the occasion of his 60th birthday

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LINEAR EQUATIONS AND THE KRONECKER PRODUCT IN COQUATERNIONS

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Abstract. We present the solution of the general linear system in n coquaternionic variables and m equations, and derive the Kronecker product from the matrix system $\mathbf{AXB}$ in coquaternions. It is shown, that the coquaternionic Kronecker product can be computed by means of dyadic coquaternionic products. The special matrix case $\mathbf{Ax} = \mathbf{b}$ is also included. The one dimensional case, including Sylvester's equation is solved and there are several, nontrivial, numerical examples.

Key words: Coquaternions, split quaternions, linear systems of equations in coquaternions, $\mathbf{Ax} = \mathbf{b}$ in coquaternions, Kronecker product in coquaternions, dyadic coquaternionic products, Sylvester's equation in coquaternions.

AMS Subject classification: 11R52, 12E15, 12Y05, 15A06, 15A24.

1. Introduction. Coquaternions were introduced, 1849, by Sir James Cockle (1819–1895), [3], [4], as complex matrices of the form

$$C := \begin{bmatrix} w & z \\ \overline{z} & \overline{w} \end{bmatrix}, \tag{1.1}$$

where the bar $\overline{}$ indicates the complex conjugate of the quantity under the bar. The matrix C is very similar to a quaternion (invented by Hamilton, 1843), which has the complex representation (see v. d. Waerden, [16], p. 55)

$$Q := \begin{bmatrix} w & z \\ -\overline{z} & \overline{w} \end{bmatrix}.$$

In both cases, the products C_1C_2, Q_1Q_2 of two coquaternions C_1, C_2, and of two quaternions Q_1, Q_2 form a coquaternion, quaternion, respectively. Thus, both sets of matrices form a real algebra. This, in particular means that the *center* is in both cases the set of matrices $\alpha\mathbf{I}, \alpha \in \mathbb{R}$, where the center is the set of elements which commute with all elements and $\mathbf{I}$ is the 2×2 identity matrix. There is one decisive difference between C and Q. The inverse of C is

$$C^{-1} := \frac{1}{|w|^2 - |z|^2} \begin{bmatrix} \overline{w} & -z \\ -\overline{z} & w \end{bmatrix}$$

and the inverse of Q is

$$Q^{-1} := \frac{1}{|w|^2 + |z|^2} \begin{bmatrix} \overline{w} & -z \\ \overline{z} & w \end{bmatrix}.$$

The inverse of C exists if and only if $|w|^2 - |z|^2 \neq 0$, whereas a quaternion Q has an inverse as long as $Q \neq 0$. Thus, the algebra of coquaternions has zero divisors, does not form a field, whereas the algebra of quaternions is free of zero divisors, it is a field, though not commutative.

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It is known, that a quaternion can also be represented by a real (4×4) matrix involving the four real numbers $\Re w, \Im w, \Re z, \Im z$, where $\Re$ stands for real part and $\Im$ stands for imaginary part of a complex number, see Gürlebeck and Sprössig, [7], p. 5.

THEOREM 1.1. *Put*

$$w = a_1 + a_2\mathbf{i}; \quad z = a_3 + a_4\mathbf{i}, \tag{1.2}$$

and define the matrix

$$C_4 := \begin{bmatrix} a_1 & -a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix}. \tag{1.3}$$

Then, the set of all matrices of the type C_4 forms an algebra, and this algebra is isomorphic to the algebra of coquaternions.

Proof: Let $C_4, \widetilde{C}_4$ be two matrices of type (1.3). Both have block structure,

$$C_4 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix}, \quad \mathbf{A}_1 := \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}, \mathbf{A}_2 := \begin{bmatrix} a_3 & a_4 \\ a_4 & -a_3 \end{bmatrix},$$

$$\widetilde{C}_4 = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2 & \mathbf{B}_1 \end{bmatrix}, \quad \mathbf{B}_1 := \begin{bmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{bmatrix}, \mathbf{B}_2 := \begin{bmatrix} b_3 & b_4 \\ b_4 & -b_3 \end{bmatrix},$$

hence,

$$C_4\widetilde{C}_4 = \begin{bmatrix} \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 & \mathbf{A}_1\mathbf{B}_2 + \mathbf{A}_2\mathbf{B}_1 \\ \mathbf{A}_1\mathbf{B}_2 + \mathbf{A}_2\mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 \end{bmatrix},$$

$$\mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 = \begin{bmatrix} a_1b_1 - a_2b_2 + a_3b_3 + a_4b_4 & -a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3 \\ a_2b_1 + a_1b_2 + a_4b_3 - a_3b_4 & -a_2b_2 + a_1b_1 + a_4b_4 + a_3b_3 \end{bmatrix},$$

$$\mathbf{A}_1\mathbf{B}_2 + \mathbf{A}_2\mathbf{B}_1 = \begin{bmatrix} a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 & a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 \\ a_2b_3 + a_1b_4 + a_4b_1 - a_3b_2 & a_2b_4 - a_1b_3 - a_4b_2 - a_3b_1 \end{bmatrix}.$$

Thus, the product has the same structure as the matrix given in (1.3). If we compare the product of two coquaternions in the form given in (1.1) with the product $C_4\widetilde{C}_4$ where (1.2) is used, then, we see, that the products are the same. □

Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$ as in (1.2). The algebra of coquaternions is also isomorphic to the algebra of all real 2×2 matrices (see Lam, p. 52, [12]):

$$\begin{aligned} C_2 &:= a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + a_4 & a_2 + a_3 \\ -a_2 + a_3 & a_1 - a_4 \end{bmatrix} =: \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}. \end{aligned} \tag{1.4}$$

Given the four matrix elements $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbb{R}$, the four components of a can be recovered by

$$a_1 = \frac{1}{2}(c_{11} + c_{22}),\ a_2 = \frac{1}{2}(c_{12} - c_{21}),\ a_3 = \frac{1}{2}(c_{12} + c_{21}),\ a_4 = \frac{1}{2}(c_{11} - c_{22}).$$

The inverse of C_2 and of C_4 (given in (1.3)) can be computed easily by applying the formula (2.6) for a^{-1}. If we denote the four basis elements in the order of the equation (1.4) by **E,I,J,K**, then they obey the same multiplication rules as $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$,

respectively, given in Table 2.1 of the next section. An algebra of this type, is also called a *split algebra*, in the current case the algebra of *split quaternions*, Lam, p. 58, [12].

That coquaternions nowadays are still useful, e. g. in physics is shown by Brody and Graefe, 2011, [2]. That paper also contains an overview over relevant properties of coquaternions and 42 references are quoted. There is another, very subtle investigation mainly on the analysis of coquaternions with application to physics by Frenkel and Libine, [6].

We will denote the set of real numbers, the set of complex numbers, the set of quaternions, the set of integers, the set of positive integers by $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{Z}, \mathbb{N}$, respectively.

2. Coquaternions. In view of the preceding section, coquaternions may be regarded as elements of $\mathbb{R}^4$ of the form

$$a := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}, \quad a_1, a_2, a_3, a_4 \in \mathbb{R}$$

which we also abbreviate by $a = (a_1, a_2, a_3, a_4)$ and which obey the multiplication rules given in Table 2.1:

Table 2.1. Multiplication table for coquaternions, the red figures differ in sign from the corresponding table for quaternions.

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

.

The algebra of coquaternions will be abbreviated by $\mathbb{H}_{\text{coq}}$. As elements of $\mathbb{R}^4$ we have $1 := (1,0,0,0), \mathbf{i} := (0,1,0,0), \mathbf{j} := (0,0,1,0), \mathbf{k} := (0,0,0,1)$, which we will abbreviate also by $\mathbb{1}_1,\ \mathbb{1}_2,\ \mathbb{1}_3,\ \mathbb{1}_4$, respectively. Let $a = (a_1, a_2, a_3, a_4),\ b = (b_1, b_2, b_3, b_4)$. For future purposes we note the following multiplication results:

$$\begin{aligned}
ab = a\mathbb{1}_1 b = \quad & a_1b_1 - a_2b_2 + a_3b_3 + a_4b_4 \\
& +(a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3)\mathbf{i} \\
& +(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)\mathbf{j} \\
& +(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)\mathbf{k},
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
a\mathbf{i}b = a\mathbb{1}_2 b = \quad & -a_2b_1 - a_1b_2 + a_4b_3 - a_3b_4 \\
& +(-a_2b_2 + a_1b_1 - a_4b_4 - a_3b_3)\mathbf{i} \\
& +(-a_2b_3 - a_1b_4 + a_4b_1 - a_3b_2)\mathbf{j} \\
& +(-a_2b_4 + a_1b_3 - a_4b_2 - a_3b_1)\mathbf{k},
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
a\mathbf{j}b = a\mathbb{1}_3 b = \quad & a_3b_1 - a_4b_2 + a_1b_3 + a_2b_4 \\
& +(a_3b_2 + a_4b_1 - a_1b_4 + a_2b_3)\mathbf{i} \\
& +(a_3b_3 - a_4b_4 + a_1b_1 + a_2b_2)\mathbf{j} \\
& +(a_3b_4 + a_4b_3 - a_1b_2 + a_2b_1)\mathbf{k},
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
a\mathbf{k}b = a\mathbb{1}_4 b = \quad & a_4b_1 + a_3b_2 - a_2b_3 + a_1b_4 \\
& +(a_4b_2 - a_3b_1 + a_2b_4 + a_1b_3)\mathbf{i} \\
& +(a_4b_3 + a_3b_4 - a_2b_1 + a_1b_2)\mathbf{j} \\
& +(a_4b_4 - a_3b_3 + a_2b_2 + a_1b_1)\mathbf{k}.
\end{aligned} \tag{2.4}$$

A coquaternion of the form $a = (a_1, 0, 0, 0)$ will be called *real* and will also be abbreviated as a_1. As we see from (2.1) (and former considerations), real coquaternions commute with all coquaternions. Let $a = (a_1, a_2, a_3, a_4)$ be a coquaternion. The first component, a_1, of a, will be denoted by $\Re(a)$ and called *real part* of a. We define the *conjugate* of a in the notation $\overline{a}$ or in the notation $\mathrm{conj}(a)$ and abs_2 of a by

$$\overline{a} := (a_1, -a_2, -a_3, -a_4), \quad \mathrm{abs}_2(a) := a_1^2 + a_2^2 - a_3^2 - a_4^2. \tag{2.5}$$

The quantity abs_2 may be negative, it is not the square of a norm. Let b be another coquaternion. There are the following rules:

$$a\overline{a} = \overline{a}a = \mathrm{abs}_2(a),\ \mathrm{abs}_2(ab) = \mathrm{abs}_2(ba) = \mathrm{abs}_2(a)\mathrm{abs}_2(b),$$

$$\overline{(ab)} = \overline{b}\,\overline{a},\ \Re(ab) = \Re(ba).$$

The coquaternion a will be called *singular* if $\mathrm{abs}_2(a) = 0$. If a is nonsingular (= not singular = invertible), then

$$aa^{-1} = a^{-1}a = (1, 0, 0, 0) \text{ holds for } a^{-1} = \frac{\overline{a}}{\mathrm{abs}_2(a)}. \tag{2.6}$$

Let the quaternion product (only in this section) be denoted by $\star$, then by comparing the coquaternion product (2.1) with the corresponding quaternion product $\star$, we see that

$$ab = a \star b + 2(a_3b_3 + a_4b_4) - 2(a_3b_4 - a_4b_3)\,\mathbf{i}.$$

The third and fourth component of ab and of $a \star b$ coincide. Thus, the coquaternions contain the complex numbers as subalgebra.

3. Linear mappings over $\mathbb{R}$ in general. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping over $\mathbb{R}$. Then it is known that such a mapping can be represented by a real matrix of size $(m \times n)$. See Horn and Johnson, p. 5, [9]. In order to find this matrix, which we will denote, in this paper, by $\mathbf{M}$, we define a *column operator* col by

$$\mathrm{col}(x) := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $x_1, x_2, \ldots, x_n$ are the components of x. If it happens that x is a matrix, we put the columns of that matrix from the left to the right into one column in order to define col for that matrix. By evaluating L at $x \in \mathbb{R}^n$ and applying the col operator we obtain

$$\mathrm{col}(L(x)) = \mathbf{M}\mathrm{col}(x), \quad \mathbf{M} \in \mathbb{R}^{m\times n}, \mathrm{col}(x) \in \mathbb{R}^n, \tag{3.1}$$

where $\mathbf{M}$ is unknown so far. Let e_j be the standard unit vectors in $\mathbb{R}^n$, $j = 1, 2, \ldots, n$. If we put $x := e_j$ we obtain

$$\mathrm{col}(L(e_j)) = \mathbf{M}\mathrm{col}(e_j) = \begin{bmatrix} \mu_{1j} \\ \mu_{2j} \\ \vdots \\ \mu_{mj} \end{bmatrix}, \quad j = 1, \ldots, n\,, \tag{3.2}$$

where μ_{ij} are the elements of the matrix $\mathbf{M}$ and the right hand side of (3.2) represents the jth column of $\mathbf{M}$. Hence, the matrix $\mathbf{M}$ is completely known by the n values $L(e_j)$. We note, that $\mathbf{M}$ will be integer if the values of L are integer.

A typical, nontrivial example is the mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$L(\mathbf{X}) = \mathbf{AXB}, \quad \mathbf{A} \in \mathbb{R}^{p\times q},\ \mathbf{B} \in \mathbb{R}^{r\times s},\ \mathbf{X} \in \mathbb{R}^{q\times r}, \tag{3.3}$$

where in this case we have $m = ps$, $n = qr$. In order to find the corresponding matrix we have to set the jth element of $\mathbf{X}$ (counted columnwise from the left) to be one and all other elements of $\mathbf{X}$ to be zero, $j = 1, 2, \ldots, qr$, and compute the corresponding $\mathrm{col}(L(\mathbf{X}))$. This gives the jth column of the wanted matrix $\mathbf{M}$. What comes out is the well known Kronecker (or tensor) product $\mathbf{M} = \mathbf{B}^{\mathrm{T}} \otimes \mathbf{A}$, a matrix of size $(ps \times qr)$. The notation $\mathbf{B}^{\mathrm{T}}$ stands for the *transposed matrix* of $\mathbf{B}$. For details see Horn and Johnson, pp. 242, 254, [8].

4. Linear systems in coquaternions. Linear equations in coquaternions are formally similar to linear equations in quaternions. An investigation of linear equations in quaternions exists by the current authors, [10], [11] and by Niven, [13]. A linear system will always be a linear system over $\mathbb{R}$. Linearity with respect to $\mathbb{C}$ or to $\mathbb{H}_{\mathrm{coq}}$ is in general not granted. A linear system in n coquaternions x_k, $k = 1, 2, \ldots, n$, and m equations will be defined as follows: Let

$$l_j(u) := \sum_{k=1}^{K_j} a_k^{(j)} u b_k^{(j)}, \quad j = 1, 2, \ldots, mn \tag{4.1}$$

be an arbitrary set of mn linear, coquaternionic mappings in one coquaternionic variable u, where $a_k^{(j)}, b_k^{(j)}$, $k = 1, 2, \ldots, K_j$ are given coquaternions, and K_j are given, positive integers, $j = 1, 2, \ldots, mn$. A system in n coquaternionic variables x_k, $k = 1, 2, \ldots, n$ and m equations will then be defined by

$$\begin{aligned} L_1(x) &:= \sum_{j=1}^{n} l_j(x_j), \\ L_2(x) &:= \sum_{j=1}^{n} l_{j+n}(x_j), \\ &\vdots \\ L_m(x) &:= \sum_{j=1}^{n} l_{j+(m-1)n}(x_j), \end{aligned}$$

where $x \in \mathbb{R}^{4n}$ consists of one column composed out of $x_1, x_2, \ldots, x_n$. The mapping $L : \mathbb{R}^{4n} \to \mathbb{R}^{4m}$ will finally be defined by

$$L(x) := \begin{bmatrix} L_1(x) \\ L_2(x) \\ \vdots \\ L_m(x) \end{bmatrix}. \tag{4.2}$$

L is not a linear mapping over $\mathbb{C}$ or $\mathbb{H}_{\mathrm{coq}}$. However, L is a linear mapping over $\mathbb{R}$ because real coquaternions commute with arbitrary coquaternions. Hence, a matrix $\mathbf{M}$ as in (3.1), (3.2) exists.

In order to find the matrix $\mathbf{M}$ which represents L, we put $x = e_j$, where the e_j represent the standard unit vectors in $\mathbb{R}^{4n}$, $j = 1, 2, \ldots, 4n$. Then $\mathrm{col}(L(e_j))$ is the jth column of the wanted matrix. The next lemma will be useful.

LEMMA 4.1. *Let $1 \le j \le 4n$ be given. Then, the following equation including its restrictions*

$$4(\kappa - 1) + r = j, \quad 1 \le \kappa \le n, \quad 1 \le r \le 4 \tag{4.3}$$

determines κ and r uniquely by

$$0 \le 4\kappa - j \le 3, \quad r = j - 4(\kappa - 1). \tag{4.4}$$

Proof: If we solve (4.3) for r we obtain $r = j - 4(\kappa - 1) = j - 4\kappa + 4$ and the given restriction for r implies the first part of (4.4) and the first part admits exactly one solution κ with the restriction given in (4.3). For this reason, the second equation in (4.4) (which coincides with the first equation in (4.3)) also has a unique solution. □

COROLLARY 4.2. *Let L be given as in (4.2). Then the jth column $(1 \le j \le 4n)$ of the matrix $\mathbf{M}$ representing L is*

$$L(e_j) := \begin{bmatrix} L_1(e_j) = l_\kappa(\mathscr{V}_r) \\ L_2(e_j) = l_{\kappa+n}(\mathscr{V}_r) \\ \vdots \\ L_m(e_j) = l_{\kappa+(m-1)n}(\mathscr{V}_r) \end{bmatrix} \tag{4.5}$$

where κ and r are determined by j applying Lemma 4.1. The first component of the unit vector e_j is the rth coordinate, $r \in \{1, 2, 3, 4\}$, of the κth coquaternion, $\kappa \in \{1, 2, \ldots, n\}$.

The task of solving a linear, coquaternionic system $L(x) = \gamma$, where γ consists of m given coquaternions, can be solved as follows. Compute the $(4m \times 4n)$ matrix $\mathbf{M}$ which represents L and solve the real matrix equation $\mathbf{M}\mathrm{col}(x) = \mathrm{col}(\gamma)$ by standard techniques.

EXAMPLE 4.3. Let $m = n = 3$ and

$$\begin{aligned} &l_1(x) := a_1^{(1)} x b_1^{(1)} + a_2^{(1)} x b_2^{(1)} + a_3^{(1)} x b_3^{(1)},\ l_2(x) := a_1^{(2)} x b_1^{(2)} + a_2^{(2)} x b_2^{(2)},\ l_3(x) := a_1^{(3)} x b_1^{(3)}, \\ &l_4(x) := a_1^{(4)} x b_1^{(4)},\ l_5(x) := a_1^{(5)} x b_1^{(5)} + a_2^{(5)} x b_2^{(5)},\ l_6(x) := a_1^{(6)} x b_1^{(6)} + a_2^{(6)} x b_2^{(6)} + a_3^{(6)} x b_3^{(6)}, \\ &l_7(x) := a_1^{(7)} x b_1^{(7)} + a_2^{(7)} x b_2^{(7)},\ l_8(x) := a_1^{(8)} x b_1^{(8)} + a_2^{(8)} x b_2^{(8)},\ l_9(x) := a_1^{(9)} x b_1^{(9)} + a_2^{(9)} x b_2^{(9)}, \end{aligned}$$

where the needed coefficients $a_k^{(j)}, b_k^{(j)}$ are defined in Table 4.4.

TABLE 4.4. Table of coquaternionic coefficients for l_1 to l_9

j	$a_1^{(j)}$	$b_1^{(j)}$	$a_2^{(j)}$	$b_2^{(j)}$	$a_3^{(j)}$	$b_3^{(j)}$
1	$(0, 0, -1, 1)$	$-(-1, 1, 2, 0)$	$(0, 1, -1, -1)$	$(-1, 2, 2, -1)$	$(0, -1, 1, 1)$	$(-1, 2, 2, -1)$
2	$(2, 0, 1, 0)$	$(-1, 2, 1, 1)$	$(-2, 0, 1, 0)$	$(-1, 0, 1, 2)$		
3	$(1, -1, 1, -2)$	$(0, 1, 2, 1)$				
4	$(0, 1, 0, 2)$	$(2, 1, 0, 0)$				
5	$(-1, 1, 2, 2)$	$(0, -1, 2, 1)$	$(0, 0, -1, 2)$	$(2, -1, 1, 1)$		
6	$-(2, 0, 1, -1)$	$(0, -1, 0, 1)$	$(-1, 2, 1, 0)$	$(2, 1, -2, 1)$	$(0, 1, 1, 0)$	$(0, 0, -2, 1)$
7	$-(1, 1, 1, 0)$	$(0, 1, 2, 1)$	$(0, 0, 0, 0)$	$(0, 1, -1, -2)$		
8	$-(2, 1, 1, 1)$	$(1, 1, 2, 0)$	$-(2, 0, 2, 1)$	$(2, 0, 0, -1)$		
9	$(0, 0, -2, 2)$	$(0, 1, 2, 2)$	$(-2, 0, 1, 0)$	$-(1, 1, -2, 0)$		

The entries of the Table 4.4 are randomly generated integers in $[-2, 2]$. This is the reason for the occurrence of a zero element. The matrix $\mathbf{M}$ which in this case corresponds to L, defined in (4.5), is

$$\mathbf{M} = \begin{bmatrix} 3 & -5 & -1 & 2 & 2 & -7 & -2 & 0 & 1 & -6 & 3 & 4 \\ -9 & -3 & 6 & -5 & 1 & -2 & 4 & 2 & -4 & 1 & -2 & -1 \\ 3 & 0 & -3 & 1 & -2 & 0 & 2 & 7 & 1 & 0 & 3 & -2 \\ 4 & 3 & -3 & 3 & -4 & 2 & -1 & -2 & -2 & 5 & -4 & -5 \\ -1 & -2 & -2 & 4 & 8 & 4 & 1 & 0 & -7 & -6 & 8 & 5 \\ 2 & -1 & 4 & 2 & 6 & -6 & 6 & -3 & 4 & -1 & -1 & 2 \\ 2 & 4 & 1 & -2 & -9 & 4 & -2 & 8 & 0 & 7 & -5 & -2 \\ 4 & -2 & 2 & 1 & 6 & 3 & 6 & 0 & -11 & -2 & 4 & 5 \\ -1 & 2 & -3 & 0 & -6 & -1 & -8 & 0 & 4 & 6 & -7 & -3 \\ 0 & 3 & -2 & -3 & -7 & -4 & -8 & 0 & 1 & 0 & -3 & -1 \\ -1 & 4 & -3 & -2 & -10 & -2 & -10 & -1 & -3 & 3 & -4 & 2 \\ -2 & -1 & 0 & 1 & -2 & 2 & 1 & -4 & 3 & -5 & -2 & 8 \end{bmatrix}.$$

Let $x_1 = (0, 0, 1, 0), x_2 = (-1, 0, 0, 2), x_3 = (0, 1, 1, -1)$ and $x = \mathrm{col}(x_1, x_2, x_3)$. Then,

$$\mathrm{col}(L(x)) := \mathbf{M}x = (-10, 9, 18, 3;\ -13, -12, 30, -7;\ 5, 3, 2, -21)^{\mathrm{T}}.$$

4.1. Linear systems in coquaternionic matrices. The general case of a linear system in n unknowns and m equations does also contain the matrix case

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in {\mathbb{H}_{\mathrm{coq}}}^{m\times n},\ \mathbf{b} \in {\mathbb{H}_{\mathrm{coq}}}^{m\times 1},\ \mathbf{x} \in {\mathbb{H}_{\mathrm{coq}}}^{n\times 1}, \tag{4.6}$$

but not vice versa.

Example 4.5. Let

$$\mathbf{A} := \begin{bmatrix} 1 & \mathbf{i} \\ \mathbf{j} & \mathbf{k} \end{bmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & \mathbf{j} \\ -\mathbf{i} & \mathbf{k} \end{bmatrix},\ \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & \mathbf{j} \\ \mathbf{i} & \mathbf{k} \end{bmatrix}.$$

If we multiply the second row of $\mathbf{A}$ from the right by $\mathbf{j}$ we obtain the first row. Thus, the right row rank of $\mathbf{A}$ is one, and if we multiply the first column from the left by $\mathbf{i}$ we obtain the second column. Thus, the left column rank of $\mathbf{A}$ is also one. Let us compute the right column rank of $\mathbf{A}$. To this purpose let

$$\begin{pmatrix} 1 \\ \mathbf{j} \end{pmatrix}\alpha + \begin{pmatrix} \mathbf{i} \\ \mathbf{k} \end{pmatrix}\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{H}_{\mathrm{coq}}.$$

If we multiply the first equation from the left by $\mathbf{j}$ we obtain $\mathbf{j}\alpha - \mathbf{k}\beta = 0$. If we add this equation to the second equation we obtain $2\mathbf{j}\alpha = 0$, and if we subtract this equation from the second equation we obtain $2\mathbf{k}\beta = 0$. Thus, $\alpha = \beta = 0$ and the right column rank is two. In a similar way one can show that the left row rank is also two. There is the following well known Theorem: Let $\mathbf{M} \in {\mathbb{H}_{\mathrm{coq}}}^{m\times n}$. Then, the left row rank coincides with the right column rank and the right row rank is equal to the left column rank. If $m = n$ and the right column rank of $\mathbf{M}$ is n, then $\mathbf{M}$ is nonsingular, which means that there is a matrix $\mathbf{M}^{-1}$ such that of $\mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the $(n \times n)$ identity matrix. Thus, $\mathbf{A}$ is nonsingular, whereas $\mathbf{A}^{\mathrm{T}}$ is singular. This example was also used by Zhang, [17], for quaternions.

Rather than specializing the general case to the matrix case, we directly treat the matrix case (4.6) with the means we have already derived. If we apply the col operator to (4.6) we obtain

$$\mathrm{col}(\mathbf{Ax}) = \mathbf{M}\mathrm{col}(\mathbf{x}) = \mathrm{col}(\mathbf{b}), \quad \mathbf{M} \in \mathbb{R}^{4m\times 4n} \tag{4.7}$$

and determine $\mathbf{M}$ by replacing $\mathrm{col}(\mathbf{x})$ with e_j, the jth standard unit vector in $\mathbb{R}^{4n}$, $j = 1, 2, \ldots, 4n$. Let $j = 1 \bmod 4, j \le 4n-3, k = 1, 2, 3, 4$. Then, the $(j+k-1)$th column of $\mathbf{M}$ is

$$\mathbf{M}_{(j+k-1)} = \begin{bmatrix} \mathrm{col}(a_{1,(j-1)/4+1}\mathbb{N}_k) \\ \mathrm{col}(a_{2,(j-1)/4+1}\mathbb{N}_k) \\ \vdots \\ \mathrm{col}(a_{m,(j-1)/4+1}\mathbb{N}_k) \end{bmatrix}. \tag{4.8}$$

Each entry, $\mathrm{col}(a_{\ell,(j-1)/4+1}\mathbb{N}_k)$, is a real 4×1 vector, $\ell = 1, 2, \ldots, m$, where we denote the jth column of $\mathbf{M}$ by $\mathbf{M}_{(j)}$. Thus, $\mathbf{M}$ of (4.7) is given in (4.8). For $\mathbf{A}$ from Example 4.5 we have

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

There are many further questions related to coquaternionic matrices, for instance, what kind of decompositions exist. This is a topic for another investigation.

4.2. The Kronecker product for coquaternions. The *coquaternionic Kronecker product* is the (real) matrix $\mathbf{M}$ which represents the linear mapping (over $\mathbb{R}$) $L : {\mathbb{H}_{\mathrm{coq}}}^{q\times r} \to {\mathbb{H}_{\mathrm{coq}}}^{p\times s}$ defined by

$$L(\mathbf{X}) = \mathbf{AXB}, \quad \mathbf{A} \in {\mathbb{H}_{\mathrm{coq}}}^{p\times q},\ \mathbf{B} \in {\mathbb{H}_{\mathrm{coq}}}^{r\times s},\ \mathbf{X} \in {\mathbb{H}_{\mathrm{coq}}}^{q\times r}. \tag{4.9}$$

It is denoted by $\mathbf{M}(\mathbf{A}, \mathbf{B})$, for short only $\mathbf{M}$. Since each element in $\mathbf{X}$ has 4 real entries, $\mathrm{col}(\mathbf{X}) \in \mathbb{R}^{4qr}$. Furthermore, we have $\mathrm{col}(L(\mathbf{X})) \in \mathbb{R}^{4ps}$ which implies $\mathbf{M} \in \mathbb{R}^{4ps\times 4qr}$. The coquaternionic Kronecker product differs from the standard Kronecker or tensor product which is $\mathbf{B}^{\mathrm{T}} \otimes \mathbf{A}$. See also (3.3), on p. 5 and the corresponding remarks. In order to find $\mathbf{M}$, we enumerate the elements of $\mathbf{X}$ by $\ell = 1, 2, \ldots, qr$, using the columns as ordering system such that (to mention an example) the first element of the second column will have the number $\ell = q+1$. We denote the so numbered elements of $\mathbf{X}$ by $\mathbf{X}(1), \mathbf{X}(2), \ldots, \mathbf{X}(qr)$. Such an enumeration technique is used in some programming languages in addition to the conventional enumeration $\mathbf{X}_{J,K}$, which denotes the Kth element in row J. We define the matrix $\mathbf{X}^{(\ell,k)}$ by putting

$$\mathbf{X}(\rho) = \begin{cases} \mathbb{N}_k & \text{for } \rho = \ell, \\ 0 & \text{for } \rho \ne \ell, \end{cases} \quad \ell = 1, 2, \ldots, qr,\ k = 1, 2, 3, 4. \tag{4.10}$$

In other words, $\mathbf{X}^{(\ell,k)}$ is the matrix which contains $\mathbb{N}_k$ at position ℓ, and contains otherwise zero elements. The corresponding columns of $\mathbf{M}$ are then

$$\mathbf{M}_{(4(\ell-1)+k)} := \mathrm{col}(\mathbf{AX}^{(\ell,k)}\mathbf{B}), \quad \ell = 1, 2, \ldots, qr, k = 1, 2, 3, 4. \tag{4.11}$$

We will describe an alternative technique to find $\mathbf{M}$. Let us assume that $\mathbf{X}$ contains only zeros, with the exception of the element $\mathbf{X}_{J,K}$ which contains one of the four unit elements $\mathscr{N}_k$. Then,

$$\mathbf{M}_{4((K-1)q+(J-1))+k} = \mathrm{col}(\mathbf{AXB}) = \mathrm{col}\Bigg(\begin{array}{c} \begin{array}{ccccccc} 1 & \dots & K-1 & K & K+1 & & r \end{array} \\ \hline \begin{array}{ccccccc} 0 & \dots & 0 & a_{1J} & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{2J} & 0 & \dots & 0 \\ & & & \vdots & & & \\ 0 & \dots & 0 & \vdots & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{pJ} & 0 & \dots & 0 \end{array} \end{array} \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & \dots & 0 & \dots & 0 \\ \mathscr{N}_k b_{K1} & \mathscr{N}_k b_{K2} & & \dots & \mathscr{N}_k b_{Ks} \\ 0 & \dots & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} \Bigg). \tag{4.12}$$

THEOREM 4.6. *Let $K \in [1,r], J \in [1,q], k \in [1,4]$ be fixed. See (4.9). The coquaternionic Kronecker product $\mathbf{M}(\mathbf{A},\mathbf{B})$ in column $[4((K-1)q+(J-1))+k]$ is given by the dyadic coquaternionic product* $\mathrm{col}(\mathbf{A}(:,J)(\mathscr{N}_k\mathbf{B}(K,:)))$.

Proof: Follows from the representation (4.12). □

EXAMPLE 4.7. We will treat an example of the form

$$\mathbf{C} = \mathbf{AXB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

where we use the data from Example 5.3 in [10]:

$$a_{11} = (0,2,2,0),\ a_{12} = (4,5,-1,-5),\ a_{21} = (0,2,2,-1),\ a_{22} = (-3,3,-3,2),$$

$$b_{11} = (0,4,-5,-4),\ b_{12} = (-2,2,1,-4),\ b_{21} = (-3,-5,2,-1),\ b_{22} = (4,3,-2,3),$$

$$x_{11} = (1,1,1,1),\ x_{12} = (1,2,1,2),\ x_{21} = (2,1,2,1),\ x_{22} = (2,2,2,2).$$

These data determine $\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ as

$$c_{11} = (28,-1,44,-65),\ c_{12} = (-58,-13,2,35),$$

$$c_{21} = (44,-92,132,-12),\ c_{22} = (-76,-79,32,89).$$

An application of (4.11) or of Theorem 4.6 yields the Kronecker product

$$\mathbf{M}(\mathbf{A},\mathbf{B}) =$$

$$\begin{bmatrix} -18 & 8 & -8 & 18 & 5 & 5 & -20 & 5 \\ 8 & 2 & -2 & -8 & 37 & -45 & -13 & -60 \\ 8 & 2 & -2 & -8 & -20 & 45 & 5 & 45 \\ -18 & 8 & -8 & 18 & -37 & 20 & 13 & 35 \\ -14 & 13 & -4 & 18 & -5 & -10 & -5 & 15 \\ 13 & -2 & -2 & -12 & -34 & -19 & -39 & 11 \\ 4 & 2 & -6 & -3 & 35 & 15 & 35 & -10 \\ -18 & 12 & -3 & 22 & 9 & 19 & 14 & -11 \\ -2 & 12 & -12 & 2 & 1 & -7 & -4 & -13 \\ 4 & -6 & 6 & -4 & -11 & -37 & 29 & -28 \\ 4 & -6 & 6 & -4 & 16 & 23 & -19 & 17 \\ -2 & 12 & -12 & 2 & 1 & 32 & -19 & 23 \\ 2 & 11 & -10 & 4 & -11 & 2 & -13 & -1 \\ 3 & -10 & 8 & -6 & -22 & 11 & -19 & -17 \\ 2 & -4 & 2 & -5 & 19 & -13 & 17 & 14 \\ 0 & 14 & -13 & 6 & 17 & -1 & 14 & 7 \end{bmatrix} \dots$$

$$\left.\begin{array}{rrrrrrrr}
14 & 8 & -8 & -14 & 16 & 24 & -19 & 6 \\
-4 & 6 & -6 & 4 & -46 & 10 & 34 & 25 \\
-4 & 6 & -6 & 4 & 41 & 4 & -44 & -14 \\
14 & 8 & -8 & -14 & 16 & -15 & -4 & -30 \\
15 & 6 & -13 & -11 & 16 & -5 & 10 & 6 \\
-6 & 5 & -3 & 9 & 7 & 32 & 12 & -28 \\
1 & 9 & -7 & 2 & -4 & -30 & -10 & 31 \\
17 & 3 & -10 & -13 & -12 & -2 & -17 & -2 \\
-10 & -14 & 14 & 10 & -12 & -19 & 18 & -1 \\
2 & -2 & 2 & -2 & 45 & 14 & -45 & -4 \\
2 & -2 & 2 & -2 & -42 & -19 & 48 & -1 \\
-10 & -14 & 14 & 10 & -15 & -4 & 15 & 14 \\
-13 & -12 & 17 & 6 & -9 & 2 & -3 & -4 \\
4 & 1 & -2 & -5 & 8 & -33 & 2 & 33 \\
-1 & -6 & 5 & 0 & -9 & 32 & -3 & -34 \\
-14 & -11 & 16 & 7 & 2 & 3 & 8 & -3
\end{array}\right] \in \mathbb{R}^{16\times 16}.$$

A further test shows that $\mathbf{M}\mathrm{col}(\mathbf{X}) = \mathrm{col}(\mathbf{C})$ for $\mathbf{X}$ chosen in the beginning of this example.

Another interesting example is Sylvester's equation [15, Sylvester, 1884]:

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}, \quad \mathbf{A} \in {\mathbb{H}_{\mathrm{coq}}}^{m\times m}, \mathbf{B} \in {\mathbb{H}_{\mathrm{coq}}}^{n\times n}, \mathbf{C}, \mathbf{X} \in {\mathbb{H}_{\mathrm{coq}}}^{m\times n} \tag{4.13}$$

$$\Rightarrow \mathrm{col}(\mathbf{AX} + \mathbf{XB}) = (\mathbf{M}(\mathbf{A}, \mathbf{I}) + \mathbf{M}(\mathbf{I}, \mathbf{B}))\mathrm{col}(\mathbf{X}) = \mathrm{col}(\mathbf{C}).$$

It may be that the linear function L has the form

$$L(\mathbf{X}) = \mathbf{AX}^{\mathrm{T}}\mathbf{B}, \quad \mathbf{A} \in {\mathbb{H}_{\mathrm{coq}}}^{p\times q},\ \mathbf{B} \in {\mathbb{H}_{\mathrm{coq}}}^{r\times s},\ \mathbf{X} \in {\mathbb{H}_{\mathrm{coq}}}^{r\times q}.$$

In order to find the corresponding Kronecker product for this case one has to change the order in which the matrix elements are counted. The matrix elements of $\mathbf{X}$ in formula (4.10) have to be counted row-wise.

Let $\mathbf{A}, \mathbf{B}$ be two coquaternionic matrices. It should be noted, that the relation $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ is in general not true. Let us denote by $\mathrm{conj}(\mathbf{A})$ the matrix where all elements of $\mathbf{A}$ have been changed to the corresponding conjugate element. Then, the relation $\mathrm{conj}(\mathbf{AB}) = \mathrm{conj}(\mathbf{A})\mathrm{col}(\mathbf{B})$ is in general also not true. However, $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$, is true, where $\mathbf{A}^* := \mathrm{conj}(\mathbf{A}^{\mathrm{T}})$.

It should be pointed out that the investigations of this paper are not restricted to equations in coquaternions. With the same technique one could solve linear systems of equations in *tessarines* or in *cotessarines*, algebras defined in $\mathbb{R}^4$ and also introduced by Cockle, [3]. Even more general algebras (finite dimensioal, real, associative algebras) would allow the application of the same technique. See Abian, Drozd and Kirichenko, Pierce [1, 5, 14].

5. The one dimensional case. For $n = m = 1$ the system (4.2) and (4.1) specialize to one equation

$$L(x) := \sum_{k=1}^{K} a_k x b_k, \quad \text{where } a_k, b_k, x \text{ are coquaternions.} \tag{5.1}$$

The simplest cases are

$$(i)\ L(x) = ax, \quad (ii)\ L(x) = xb, \quad (iii)\ L(x) = axb,$$

We denote the real 4×4 matrices for the three cases, respectively, by

$${}_a\mathbf{M}, \quad \mathbf{M}_b, \quad {}_a\mathbf{M}_b.$$

Let $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4)$. Then the three matrices are

$$(5.2) \qquad {}_a\mathbf{M} := (a\mathscr{N}_1, a\mathscr{N}_2, a\mathscr{N}_3, a\mathscr{N}_4) = \begin{bmatrix} a_1 & -a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix},$$

$$(5.3) \qquad \mathbf{M}_b := (\mathscr{N}_1 b, \mathscr{N}_2 b, \mathscr{N}_3 b, \mathscr{N}_4 b) = \begin{bmatrix} b_1 & -b_2 & b_3 & b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & -b_4 & b_1 & b_2 \\ b_4 & b_3 & -b_2 & b_1 \end{bmatrix},$$

$$(5.4) \qquad {}_a\mathbf{M}_b := [ab, a\mathscr{N}_2 b, a\mathscr{N}_3 b, a\mathscr{N}_4 b],$$

where the four columns are already defined in (2.1) to (2.4). Matrix ${}_a\mathbf{M}$ is identical with the matrix C_4, defined in (1.3) and this is clear because C_4 represents the coquaternion $a = (a_1, a_2, a_3, a_4)$ and the mapping L is defined by $L(x) = ax$. A real, square matrix is named *singular* if it is not invertible, or in other words if its determinant vanishes.

THEOREM 5.1. *Matrix ${}_a\mathbf{M}$ is singular if and only if a is singular. Matrix $\mathbf{M}_b$ is singular if and only if b is singular. Matrix ${}_a\mathbf{M}_b$ is singular if a or b is singular.*

Proof: Let $L(x) = ax$ and $a \neq 0$. The equation $ax = c$ has clearly a unique solution if a is nonsingular. If a is singular, we multiply by $\overline{a}$ and obtain $0 = \overline{a}ax = \overline{a}c$, which does not have a unique solution. The other cases are similar. □

EXAMPLE 5.2. We apply the technique described in Section 3 to

$$\begin{aligned} L(x) &= a_1 x b_1 + a_2 x b_2 + a_3 x b_3, \quad \text{where} \\ a_1 &= (1,1,0,1),\ b_1 = (1,-1,-1,0), \\ a_2 &= (1,-1,-1,2),\ b_2 = (2,-1,1,1), \\ a_3 &= (-1,-2,0,2),\ b_3 = (0,-1,0,0). \end{aligned}$$

From these data we obtain

$$\begin{aligned} a_1 b_1 &= (2,-1,-2,0),\ a_1 \mathbf{i} b_1 = (-1,2,2,0), \\ a_1 \mathbf{j} b_1 &= (0,0,0,1),\ a_1 \mathbf{k} b_1 = (2,-2,-3,0), \\ a_2 b_2 &= (2,0,-2,3),\ a_2 \mathbf{i} b_2 = (6,0,3,6), \\ a_2 \mathbf{j} b_2 &= (0,3,0,0),\ a_2 \mathbf{k} b_2 = (7,0,2,6), \\ a_3 b_3 &= (-2,1,-2,0),\ a_3 \mathbf{i} b_3 = (-1,-2,0,2), \\ a_3 \mathbf{j} b_3 &= (2,0,2,-1),\ a_3 \mathbf{k} b_3 = (0,-2,1,2), \end{aligned}$$

and the matrix $\mathbf{M}$ which represents the linear mapping L is

$$\mathbf{M} = \begin{bmatrix} 2 & 4 & 2 & 9 \\ 0 & 0 & 3 & -4 \\ -6 & 5 & 2 & 0 \\ 3 & 8 & 0 & 8 \end{bmatrix}.$$

In order to find the four columns of $\mathbf{M}$, one has to compute

$$\operatorname{col}\left(\sum_{j=1}^{3} a_j \mathscr{N}_k b_j\right),\ k = 1,2,3,4.$$

5.1. Sylvester's equation in coquaternions. Sylvester's equation (see [15]) in coquaternions, a special case of (4.13) and of (5.1), reads

$$L(x) := ax + xb = c, \quad a, b, c, x \in \mathbb{H}_{\text{coq}}, \tag{5.5}$$

where we will put $a := (a_1, a_2, a_3, a_4)$, $b := (b_1, b_2, b_3, b_4) \in \mathbb{H}_{\text{coq}}$. It is clear that the matrix corresponding to this system is

$$\mathbf{M} := {}_a\mathbf{M} + \mathbf{M}_b = \begin{bmatrix} s_1 & -s_2 & s_3 & s_4 \\ s_2 & s_1 & d_4 & -d_3 \\ s_3 & d_4 & s_1 & -d_2 \\ s_4 & -d_3 & d_2 & s_1 \end{bmatrix}, \tag{5.6}$$

where ${}_a\mathbf{M}, \mathbf{M}_b$ are defined in (5.2), (5.3) and where

$$s_j := a_j + b_j,\ d_j := a_j - b_j, \quad j = 1, 2, 3, 4.$$

Sylvester's equation will be called *singular* if the matrix $\mathbf{M}$ defined in (5.6) is singular. We observe that $d_1 := a_1 - b_1$ is not occurring in (5.6).

Example 5.3. Let $a = (1, 1, 1, 2)$, $b = (1, -2, 2, 1)$. Then,

$$\mathbf{M} := \begin{bmatrix} 2 & 1 & 3 & 3 \\ -1 & 2 & 1 & 1 \\ 3 & 1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}.$$

In this case a is nonsingular, b is singular, and $\mathbf{M}$ is, nevertheless, nonsingular.

Lemma 5.4. *The determinant of* $\mathbf{M}$*, defined in (5.6) is*

$$\det(\mathbf{M}) = s_1^2 \left(s_1^2 + s_2^2 + d_2^2 - (s_3^2 + d_3^2) - (s_4^2 + d_4^2)\right) + (-s_2 d_2 + s_3 d_3 + s_4 d_4)^2. \tag{5.7}$$

Proof: Apply the expansion formula to (5.6) and collect terms. □

The singularity of Sylvester's equation, expressed by the singularity of the matrix $\mathbf{M}$, is characterized in the next theorem.

Theorem 5.5. *The matrix* $\mathbf{M}$*, defined in (5.6) is singular if and only if either*

$$s_1 := a_1 + b_1 = 0 \text{ and } -\operatorname{abs}_2(a) + \operatorname{abs}_2(b) = 0, \quad \text{or } s_1 \neq 0 \text{ and}$$

$$s_1^2 \left(-d_1^2 + 2\left(\operatorname{abs}_2(a) + \operatorname{abs}_2(b)\right)\right) + \left(-\operatorname{abs}_2(a) + \operatorname{abs}_2(b) + s_1 d_1\right)^2 = 0,$$

where the definition of abs_2 *is given in (2.5).*

Proof: Use $s_j^2 + d_j^2 = 2(a_j^2 + b_j^2)$ and $s_j d_j = a_j^2 - b_j^2$, $j = 1, 2, 3, 4$ and insert this into (5.7) of Lemma 5.4. □

An example where the second part of Theorem 5.5 applies is $a = (2, 1, 1, 2)$, $b = (1, 2, 2, 5)$. In this case a is singular, b is nonsingular, but $\mathbf{M}$ is singular.

Corollary 5.6. *Let both,* $a, b \in \mathbb{H}_{\text{coq}}$ *be singular. Then,* $\mathbf{M}$ *defined in (5.6) is singular.*

Proof: The assumptions are $\operatorname{abs}_2(a) = \operatorname{abs}_2(b) = 0$. The singularity of $\mathbf{M}$ follows from one of the conditions of Theorem 5.5. □

Conversely, one cannot say, that nonsingular a, b imply a nonsingular matrix $\mathbf{M}$. A necessary, but not sufficient condition for $\mathbf{M}$ being nonsingular is, that at least a or b is nonsingular.

Example 5.7. Let $a = (a_1, a_2, a_3, a_4) = (0, 0, 1, 0) = \mathbf{j}, b = (b_1, b_2, b_3, b_4) = (0, 0, 0, 1) = \mathbf{k}$. Then, $s_1 = a_1 + b_1 = 0, \mathrm{abs}_2(a) = -1, \mathrm{abs}_2(b) = -1$, thus, both a, b are nonsingular, but the first condition of Theorem 5.5 is satisfied, and, hence $\mathbf{M}$, defined in (5.6), and Sylvester's equation (5.5) are singular.

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