QCD in the δ-Regime

W. Bietenholz[a]**, N. Cundy**[b]**, M. Göckeler**[c]**, R. Horsley**[d]**,**
Y. Nakamura[e]**, D. Pleiter**[f]**, P.E.L. Rakow**[g]**, G. Schierholz**[c,h]
and J.M. Zanotti[d]   **(QCDSF Collaboration)**

[a] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
   A.P. 70-543, C.P. 04510 México, Distrito Federal, Mexico
[b] Lattice Gauge Theory Research Center, FPRD, and CTP, Department of Physics &
   Astronomy, Seoul National University, Seoul, 151-747, South Korea
[c] Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany
[d] School of Physics, University of Edinburgh, Edinburgh EH9 3JZ, United Kingdom
[e] Center for Computational Sciences, University of Tsukuba,
   Tsukuba, Ibaraki 305-8577, Japan
[f] Deutsches Elektronen-Synchrotron DESY, 15738 Zeuthen, Germany
[g] Theoretical Physics Division, Dept. of Mathematical Sciences, University of Liverpool,
   Liverpool, L69 3BX, United Kingdom
[h] Deutsches Elektronen-Synchrotron DESY, 22603 Hamburg, Germany

E-mail: wolbi@nucleares.unam.mx

Abstract. The δ-regime of QCD is characterised by light quarks in a small spatial box, but a large extent in (Euclidean) time. In this setting a specific variant of chiral perturbation theory — the δ-expansion — applies, based on a quantum mechanical treatment of the quasi one-dimensional system. In particular, for vanishing quark masses one obtains a residual pion mass M_π^R, which has been computed to the third order in the δ-expansion. A comparison with numerical measurements of this residual mass allows for a new determination of some Low Energy Constants, which appear in the chiral Lagrangian. We first review the attempts to simulate 2-flavour QCD directly in the δ-regime. This is very tedious, but results compatible with the predictions for M_π^R have been obtained. Then we show that an extrapolation of pion masses measured in a larger volume towards the δ-regime leads to good agreement with the theoretical predictions. From those results, we also extract a value for the (controversial) sub-leading Low Energy Constant $\bar{l}_3$.

1. QCD and Chiral Perturbation Theory

The Lagrangian of QCD is scale invariant, but its quantisation singles out an intrinsic energy $\Lambda_{\rm QCD}$, which sets the scale for the hadron spectrum. Our daily life is dominated by low energy and therefore by the lightest quark flavours, *i.e.* quarks with masses $m_{\rm q} \ll \Lambda_{\rm QCD}$. In the limit of vanishing quark masses, their left- and right-handed spinor components (Ψ_L and Ψ_R) decouple. Thus the Lagrangian takes the structure

$$\mathcal{L}_{\rm QCD} = \bar{\Psi}_L D \Psi_L + \bar{\Psi}_R D \Psi_R + \mathcal{L}_{\rm gauge} \ , \tag{1}$$

where D is the Dirac operator. For N_f massless quark flavours, this Lagrangian has the global symmetry

$$U(N_f)_L \otimes U(N_f)_R = \underbrace{SU(N_f)_L \otimes SU(N_f)_R}_{\text{chiral flavour symmetry}} \otimes \underbrace{U(1)_V}_{\text{baryon number conservation}} \otimes \underbrace{U(1)_A}_{\text{axial symmetry}} . \quad (2)$$

Here we split off the phases; the (vectorial) symmetry under simultaneous left- and right-handed phase rotation, $U(1)_V$, corresponds to the conservation of the baryon number. The remaining $U(1)_A$ symmetry — for opposite L and R phase rotations — is the axial symmetry. That is a symmetry of the classical theory, which breaks explicitly under quantisation, *i.e.* it is anomalous. We are interested in the remaining *chiral flavour symmetry,* which (in infinite volume) breaks spontaneously,

$$SU(N_f)_L \otimes SU(N_f)_R \longrightarrow SU(N_f)_{L+R} . \quad (3)$$

Chiral Perturbation Theory (χPT) deals with an effective Lagrangian in term of fields in the coset space of this spontaneous symmetry breaking, $U(x) \in SU(N_f)$ [1]. Thus it captures the lightest degrees of freedom, which dominate low energy physics, in this case given by $N_f^2 - 1$ Nambu-Goldstone bosons. The effective chiral Lagrangian $\mathcal{L}_{\text{eff}}$ embraces all terms which are compatible with the symmetries. This concept also extends to the case where small quark masses are added, $m_q \gtrsim 0$. Then one deals with light pseudo Nambu-Goldstone bosons, which are identified with the light mesons; for $N_f = 3$ this includes the pions, the kaons and the η-meson.

Here we consider the case $N_f = 2$, so we only deal with the quark flavours u and d. We assume them to be degenerate, *i.e.* to have both the mass m_q. In this case the field $U(x) \in SU(2)$ describes the pion triplet. The terms in $\mathcal{L}_{\text{eff}}$ are ordered according to an energy hierarchy, which depends on the number of derivatives and powers of m_q. Some of the first terms are

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \frac{1}{4}F_\pi^2 \, \text{Tr}[\partial_\mu U^\dagger \partial_\mu U] + \frac{1}{2}\Sigma m_q \, \text{Tr}[U + U^\dagger] \\
&\quad - \frac{1}{4}l_1 \left(\text{Tr}[\partial_\mu U^\dagger \partial_\mu U]\right)^2 - \frac{1}{4}l_2 \left(\text{Tr}[\partial_\mu U^\dagger \partial_\nu U]\right)^2 \\
&\quad - (l_3 + l_4)\left(\frac{\Sigma m_q}{2F_\pi}\right)^2 \left(\text{Tr}[U + U^\dagger]\right)^2 + l_4 \frac{\Sigma m_q}{4F_\pi} \, \text{Tr}[\partial_\mu U^\dagger \partial_\nu U] \, \text{Tr}[U + U^\dagger] + \dots \quad (4)
\end{aligned}$$

Each term comes with a coefficient, which is a free parameter of the effective theory. These coefficients are known as the *Low Energy Constants* (LECs). The leading LECs are the pion decay constant F_π (which was measured as $F_\pi = 92.4$ MeV) and the chiral condensate Σ (the order parameter for chiral symmetry breaking). The l_i are sub-leading LECs. In the chiral limit $m_q = 0$, only F_π occurs in the leading order. This determines an intrinsic scale $\Lambda_\chi = 4\pi F_\pi \simeq 1.2$ GeV, which adopts the rôle of Λ_{QCD} in χPT.

The LECs are of primary importance for low energy hadron physics. To some extent they can be fixed from phenomenology. On the theoretical side, they can only be determined from QCD as the underlying fundamental theory. Since this refers to low energy, it is a non-perturbative task, and therefore a challenge for lattice simulations. If one succeeds in their determination, we arrive at a rather complete, QCD-based formalism for low energy hadron physics.

2. Pions in a finite volume: p-regime, ϵ-regime and δ-regime

For a system of pions in a finite volume, say with periodic boundary conditions and with a characteristic extent L, the low energy expansion can be formulated in terms of the dimensionless lowest non-zero momentum $p_\mu/\Lambda_\chi \sim 1/(2F_\pi L)$, and of the correlation length $\xi = M_\pi^{-1}$ (M_π is

the pion mass). Both p_μ and M_π should be light compared to Λ_χ. Depending on the size and shape of the volume, one distinguishes various regimes, with different counting rules for these expansion parameters:

- p-Regime: This is the standard setting with a large volume, $L \gg \xi$, and therefore small finite size effects [2]. In this case the counting is simply $O(1/L) = O(M_\pi)$. From the Lagrangian we can read off (on tree level) the Gell-Mann–Oakes–Renner relation

$$M_\pi^2 = \frac{\Sigma}{F_\pi^2} \, m_{\rm q} \; . \tag{5}$$

- ϵ-Regime: This regime refers to a small box in Euclidean space [3], say $V = L^4$ with $L \lesssim \xi$ (where ξ is still the would-be inverse pion mass in a large volume, with the same quark mass; in Nature $\xi \simeq 1.5$ fm). In this regime, the χPT counting rules read $O(1/L) = O(m_{\rm q}) = O(M_\pi^2/\Lambda_\chi)$, unlike the p-regime counting.
  Experimentally the ϵ-regime is not accessible, but QCD simulations in (or at least close to) this regime are feasible. They are of interest in particular because the LECs determined in this regime are the same that occur in large volume. Therefore we can extract physical information even from an unphysical regime. Since this can be achieved with a modest lattice size, this method is attractive from a practical perspective [4]. It has been intensively explored since 2003. It is difficult to simulate safely inside this regime, but certain properties, which are characteristic for the ϵ-regime, have been observed. Ref. [5] provides a short overview.

- δ-Regime: Here one deals with a small *spatial* volume, but a large extent T in Euclidean time [6], say

$$L^3 \times T \quad , \qquad L \lesssim \xi \ll T \; . \tag{6}$$

  This relation for L and T is depicted in Fig. 1 on the left (it is exactly opposite to the setting used in studies of QCD at finite temperature). The counting rules for the corresponding *δ-expansion* are

$$\frac{1}{\Lambda_\chi L} = O(\delta) \quad , \qquad \frac{M_\pi}{\Lambda_\chi} \, , \, \frac{1}{\Lambda_\chi T} = O(\delta^3) \; . \tag{7}$$

A map of these three regimes in terms of the pion mass and the inverse extent in Euclidean time (the temperature) is shown in Fig. 1 on the right.

This article addresses the δ-regime. It is far less known and explored than the p- and the ϵ-regime, but it shares with the latter the exciting property that physical LECs can be extracted from an unphysical setting. A further motivation for studying QCD in a "δ-box" is that its shape allows (approximately) for a simplified analytical treatment in terms of 1d field theory, *i.e. quantum mechanics.* In this case one considers a quantum rotator as described by the 1d $O(4)$ model, due to the local isomorphism between the orthogonal group $O(4)$ and the chiral symmetry group $SU(2)_L \otimes SU(2)_R$. Closely related systems have applications in solid state physics, in particular regarding quantum anti-ferromagnets [7, 8].

Spontaneous symmetry breaking does not occur in a finite volume. Therefore the pions[1] — *i.e.* the pseudo Nambu-Goldstone bosons — cannot become massless in the chiral limit $m_{\rm q} = 0$, in contrast to the infinite volume. We may consider a fixed box of a δ-shape and vary the quark mass: a large value of $m_{\rm q}$ implies a large pion mass, so that we enter the p-regime and the Gell-Mann–Oakes–Renner relation (5), $m_q \propto M_\pi^2$, is approximated. For small $m_{\rm q}$ the pion mass turns into a plateau, which ends in the chiral limit at a *residual pion mass* M_π^R. This behaviour is illustrated schematically in Fig. 2.

---

[1] One might argue if the term "pion" is adequate in the δ-regime. We find it acceptable and convenient, but readers who disagree may simply denote M_π^R (see below) as the "mass gap".
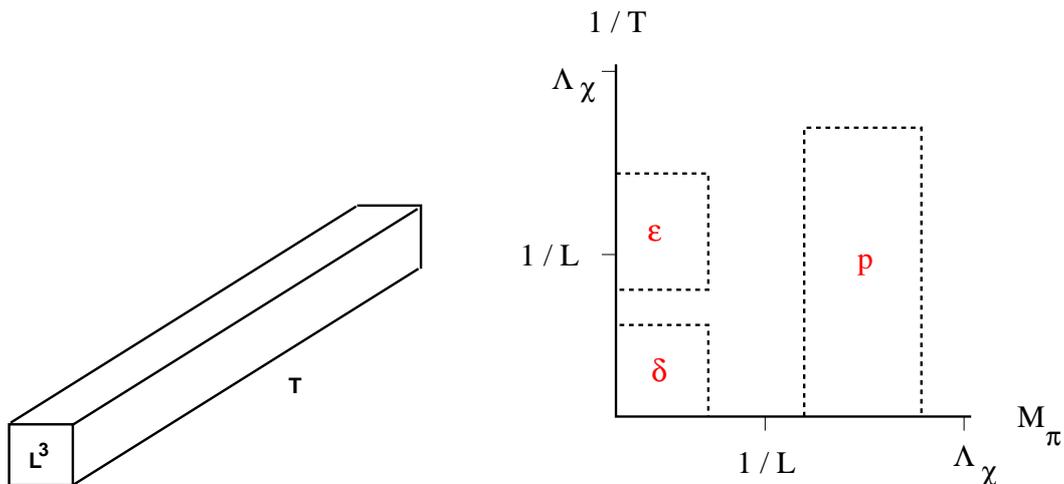
Figure 1. On the left: an illustration of a typical shape of a δ-box, *i.e.* an anisotropic finite volume where a pion gas can be treated by the δ-expansion. On the right: a schematic map of the applicability domains of three different expansion rules of χPT, namely the *p*- , the ϵ- and the δ-regime. The dashed lines indicate regions where clearly one expansion holds; in the transition zones between these regions various expansions could work more or less.
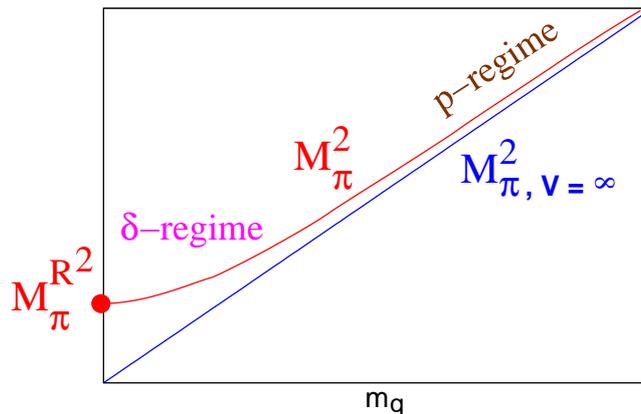

Figure 2. A qualitative picture of the expected behaviour of the pion mass squared in a δ-box. For heavy quarks and pions we approximate the *p*-regime relation $m_q \propto M_\pi^2$. For light quarks the pion mass attains a plateau, and finally (in the chiral limit $m_q = 0$) the residual value M_π^R.

The value of M_π^R can be computed with the δ-expansion. The spectrum of the $O(4)$ quantum rotator (a quantum mechanical particle on the sphere S^3) is given by $E_\ell = \ell(\ell+2)/(2\Theta)$, so the mass gap amounts to $M_\pi^R = 3/(2\Theta)$. The challenge is now to compute the moment of inertia Θ. In his seminal paper on the δ-regime, H. Leutwyler gave its value to leading order (LO) as $\Theta \approx F_\pi^2 L^3$. Thus the residual pion mass can be written as

$$M_\pi^R = \frac{3}{2F_\pi^2 L^3 (1 + \Delta)} \; . \tag{8}$$

The shift Δ captures higher order corrections, which are suppressed in powers of $1/(F_\pi L)^2$. They have been evaluated to next-to-leading order (NLO) in Ref. [8], and recently even to

next-to-next-to-leading order (NNLO) [9], which yields

$$\Delta = \frac{0.4516\dots}{F_\pi^2 L^2} + \frac{0.08843\dots}{F_\pi^4 L^4}\Big[1 - 0.1599\dots\Big(\ln(\Lambda_1 L) + 4\ln(\Lambda_2 L)\Big)\Big] \ . \tag{9}$$

Λ_i are scale parameters for the sub-leading LECs. The latter are given at the scale of the physical pion mass as

$$\bar{l}_i = \ln(\Lambda_i/M_\pi^{\text{phys}})^2 \ . \tag{10}$$

Even more recent papers addressed again the NNLO of the δ-expansion [10], and the corrections due to finite m_{q} [11]. In the following we will discuss numerical results for M_π^R. We see that a confrontation with the analytical prediction in eqs. (8), (9) could enable a new determination of a set of LECs from first principles of QCD.

3. Attempts to simulate QCD in the δ-regime

The straight way to measure M_π^R are simulations directly *in* the δ-regime. Since the δ-box differs from the lattice shapes in usual simulations, this requires the special purpose generation of configurations. Moreover, precise chirality is vital in this regime, hence one is supposed to use a formulation of lattice quarks which preserves chiral symmetry. Such lattice fermions are known since the late 90ies, but their simulation is extremely tedious, in particular with dynamical quarks (*i.e.* keeping track of the fermion determinant in the generation of gauge configurations). We anticipate that so far there are no robust results of simulations clearly inside the δ-regime. In this section we summarise the efforts that have been carried out so far.

At the Symposium LATTICE2005 D. Hierl presented a first attempt to simulate 2-flavour QCD in the δ-regime [12]. That study used a truncated version of a chiral lattice Dirac operator, so a first question is if the quality of approximate chirality was sufficient for that purpose. Since this Dirac operator is very complicated, that simulation was performed with a non-standard algorithm, which probes the fermion determinant with a stochastic estimator. The spatial volume was $\approx (1.2 \text{ fm})^3$, and the results for M_π at small quark masses agreed well with the LO of the δ-expansion, *i.e.* eq. (8) at $\Delta = 0$ (and with the phenomenological value of F_π).

In 2007 the QCDSF Collaboration generated a new set of data, which have not been published. They were obtained with dynamical overlap quarks, which are exactly chiral. This simulation used the Hybrid Monte Carlo algorithm (*i.e.* the reliable standard algorithm). The lattice had a modest size of $8^3 \times 16$ sites, and the spatial box length was again $L \approx 1.2$ fm. At first sight the results seemed to look fine: for decreasing m_{q} we saw a transition from a Gell-Mann–Oakes–Renner type behaviour towards a plateau. Its value agreed with the chirally extrapolated value of Ref. [12], and therefore also with the LO of eq. (8).

Unfortunately, this is *not* the end of the story. If we proceed to the NLO correction, *i.e.* if we include the first term of Δ given in eq. (9), the predicted value for M_π^R decreases drastically in this small volume — from 782.5 MeV down to 321.7 MeV — and the agreement with the above data is gone. Considering this dramatic effect of the NLO correction one might worry that the δ-expansion could converge only very slowly in this small box, and such simulations are not instructive at all. However, adding also the NNLO correction alters the NLO result only a little — to (336.3 ± 7.6) MeV — so it is reasonable to assume the δ-expansion to be already well converged.[2]

Thus simulations in this small box *can* be useful in view of a confrontation with the analytical predictions for M_π^R. In fact a second sequence of runs by the QCDSF Collaboration (performed

---

[2] To obtain this theoretically predicted value, we inserted the LECs as far as they are known. In particular the sub-leading LECs $\bar{l}_1 = -0.4 \pm 0.6$ and $\bar{l}_2 = 4.1 \pm 0.1$ are taken from Ref. [13], along with their uncertainties, which imply the uncertainty in the NNLO value of M_π^R.

m_{q}	a [fm]	M_π	M_π^R (NNLO)
0.008	0.102(9)	(535.8 ± 54.5) MeV	(510.3 ± 30.7) MeV
0.01	0.104(8)	(580.5 ± 47.2) MeV	(505.8 ± 30.3) MeV
0.03	0.114(8)	(444.8 ± 31.4) MeV	(476.2 ± 33.4) MeV
0.04	0.111(8)	(476.4 ± 34.7) MeV	(486.2 ± 32.6) MeV
0.05	0.111(9)	(511.9 ± 43.9) MeV	(486.2 ± 35.0) MeV

Table 1. The results by QCDSF Collaboration of the year 2008, on a $8^3 \times 16$ lattice with 5 values of the mass m_{q} for the two degenerate dynamical overlap quark flavours. We display the measured values of the physical lattice spacing a and the pion mass, as well as the NNLO prediction for the residual pion mass M_π^R. (The errors capture the uncertainty in a, in the pion mass in lattice units, and in the LECs $\bar{l}_1$ and $\bar{l}_2$, cf. eq. (14).)

in 2008) yielded data much closer to the NNLO prediction. They involved 5 quark masses; the results for the physical lattice spacing a and the pion mass are given in Table 1. Since a varies for the different values of m_{q}, the box length $L = 8a$ and the prediction for M_π^R vary as well. In Fig. 3 we compare the numerically measured pion masses and the corresponding NNLO δ-expansion results. They are compatible, in contrast to the earlier data, which were probably not well thermalised. In these new runs thermalisation is accomplished, but the ratio $T/L = 2$ is still modest. We therefore proceeded to a $8^3 \times 32$ lattice, where our runs are ongoing. Still, the quality of agreement with the δ-expansion that we observed already on the $8^3 \times 16$ lattice is impressive.
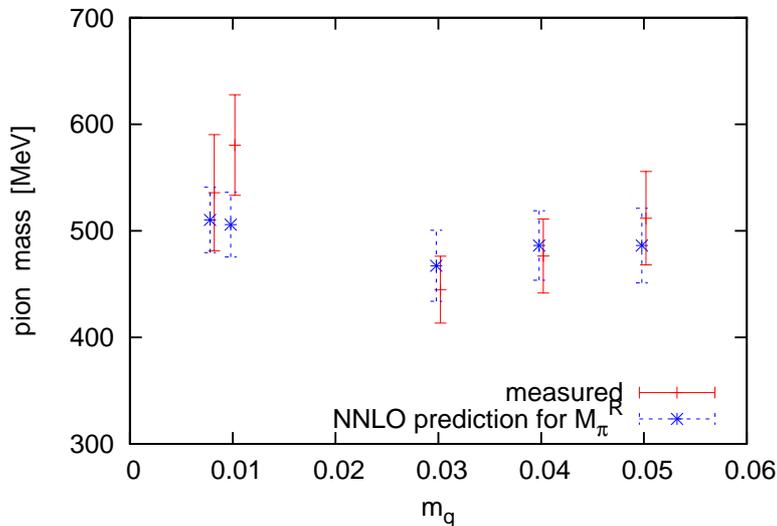

Figure 3. An illustration of our results given in Table 1: we show the numerically measured pion mass and the theoretical residual mass M_π^R, as predicted by the NNLO δ-expansion. (The latter actually refer to the chiral limit, but its value depends on L and therefore on the lattice spacing a, which varies for the simulations at different m_{q}.) We observe good agreement between the measured and the predicted values.

4. Residual pion mass by an extrapolation from the p-regime

In this section we proceed to a different approach. It is based on simulation results in the p-regime (up to the transition zone), which are then extrapolated towards the δ-regime. Details of this study are given in Ref. [14]. Also in this framework we consider it essential to use dynamical

quarks, but we do not insist on exact chiral symmetry in the p-regime. Hence we used Wilson fermions, which is an established standard lattice fermion formulation, in a form which corrects $O(a)$ lattice artifacts. Thus the simulation was much faster, and we could tackle much larger lattices than those mentioned in the Section 3; our data reported below were obtained on three lattice sizes: $24^3 \times 48$, $32^3 \times 64$ and $40^3 \times 64$. On the other hand, this lattice regularisation breaks the chiral symmetry explicitly, so that additive mass renormalisation sets in. Nevertheless we were able to attain very light pion masses.

For the gauge part we used the standard plaquette lattice action. Our simulations were carried out at two values for the strong gauge coupling g_s, respectively the parameter $\beta = 6/g_s^2$. We determined the physical lattice spacing a from the measured nucleon mass, which revealed that we were dealing with fine lattices,

$$\beta = 5.29 \;\rightarrow\; a \simeq 0.075 \text{ fm} \quad , \quad \beta = 5.4 \;\rightarrow\; a \simeq 0.067 \text{ fm} \quad . \tag{11}$$

Thus the spatial size was in the range $L \simeq 1.6 \text{ fm} \ldots 3.0 \text{ fm}$. Due to the additive mass renormalisation, we could not refer to the bare quark mass anymore. We measured the current quark mass by means of the PCAC relation,

$$m_q = \frac{\langle \partial_4 A_4(\vec{0}, x_4) P(0) \rangle}{\langle P(\vec{0}, x_4) P(0) \rangle} \;, \tag{12}$$

where P is the pseudoscalar density, and A_μ is the axial current. We observed practically no finite size effects on a and on m_q, but we did see a striking L-dependence of M_π, as expected. These quantities were found in the range

$$m_q = 3.60 \text{ MeV} \ldots 231 \text{ MeV} \quad , \quad M_\pi = 174 \text{ MeV} \ldots 1.52 \text{ GeV} \quad , \quad M_\pi L = 2.7 \ldots 9.7 \;. \tag{13}$$

The latter confirms that our data range from the deep p-regime to the transition zone. A few missing data points have been completed with a (lengthy) formula for exponentially suppressed finite size effects [15, 16] within the p-regime, which holds up to $O(p^4)$. This formula involves the renormalised sub-leading LECs $\bar{l}_i$, $i = 1 \ldots 4$, see eqs. (4) and (10). Well established phenomenological values were composed in Ref. [13] ($\bar{l}_1$, $\bar{l}_2$ were anticipated in footnote 2),

$$\bar{l}_1 = -0.4 \pm 0.6 \quad , \quad \bar{l}_2 = 4.1 \pm 0.1 \quad , \quad \bar{l}_3 = 2.9 \pm 2.4 \quad , \quad \bar{l}_4 = 4.4 \pm 0.4 \quad . \tag{14}$$

They were estimated from $\pi\pi$ scattering data, and in particular the $\bar{l}_4$ value is based on the scalar pion form factor. The dark horse in this context is $\bar{l}_3$: we replaced the above value by $\bar{l}_3 \approx 4.2$, which we obtained in our study, see below.

Our measured and interpolated data are given in Ref. [14]. We extrapolated them towards the δ-regime and extracted in particular a value for M_π^R based on the relatively simple chiral extrapolation formula

$$M_\pi(L)^2 = M_\pi^{R\,2} + C_1 m_q \left[1 + C_2 m_q \ln(C_3 m_q) \right] \;, \tag{15}$$

which interpolates between the $O(p^4)$ correction formula (for large LM_π) and the chiral limit [13]. The C_i and M_π^R are treated as free parameters to be fixed by the fit. Our data and the fits for $\beta = 5.29$ and for $\beta = 5.4$ are shown in Figs. 4 and 5, respectively. These plots also illustrate the extrapolation result for M_π^R in the chiral limit.

Our main result is shown in Fig. 6. It compares the extrapolated values for M_π^R with the predictions based on the δ-expansion to LO, NLO and NNLO, as a function of L. The latter two predictions are very close to each other for the volumes under consideration, so we can again assume the expansion to be well converged.
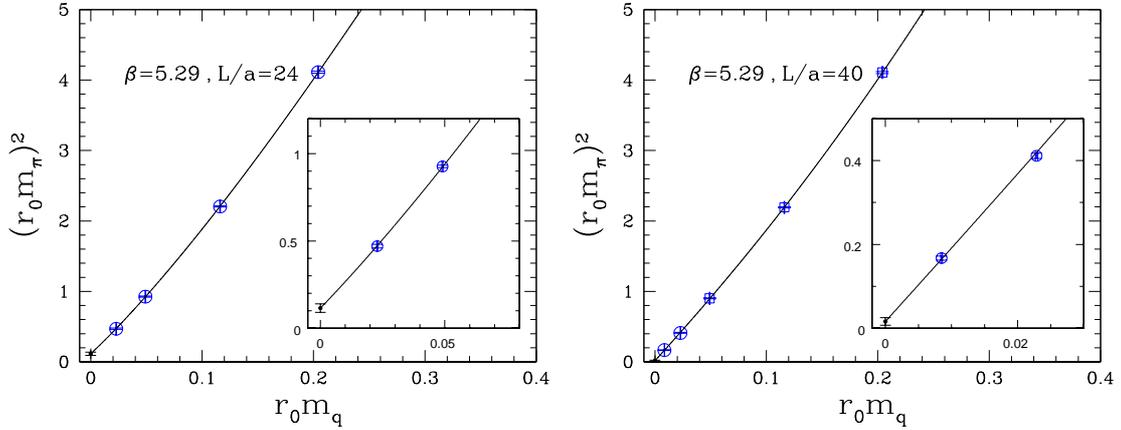
Figure 4. Our chiral extrapolation referring to the lattice spacing $a \simeq 0.075$ fm (corresponding to $\beta = 5.29$) with $L \simeq 1.8$ fm (on the left) and $L \simeq 3.0$ fm (on the right). The data points show the measured pion mass squared against the quark mass in the p-regime (the Sommer scale parameter $r_0 = 0.467$ fm is employed to convert them into dimensionless units). The curve is the fit according to eq. (15). It ends in the chiral limit, where we illustrate the extrapolated value for M_π^R and its error.
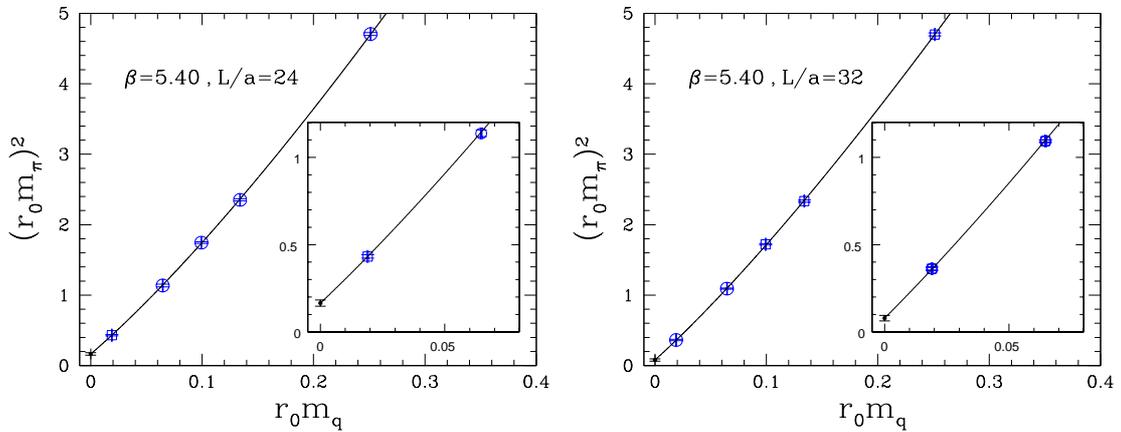
Figure 5. The same as Fig. 4, but now for the finer lattices with $a \simeq 0.067$ fm (corresponding to $\beta = 5.4$), such that $L \simeq 1.6$ fm (on the left) and $L \simeq 2.1$ fm (on the right).

The extrapolation results for M_π^R reach down to values even below the physical pion mass. The plot also shows the line where the product $M_\pi L$ decreases below 1; this can be roughly considered as the boundary of the δ-regime. Our extrapolations lead close to this boundary, and in one case *into* the δ-regime.[3]

In particular this plot shows that the extrapolated masses M_π^R match this curve remarkably well. It would not have been obvious to predict this feature, because our data were obtained in

---

[3] In that case, the extrapolation should actually turn into a plateau at tiny m_q, but this would hardly change the result.
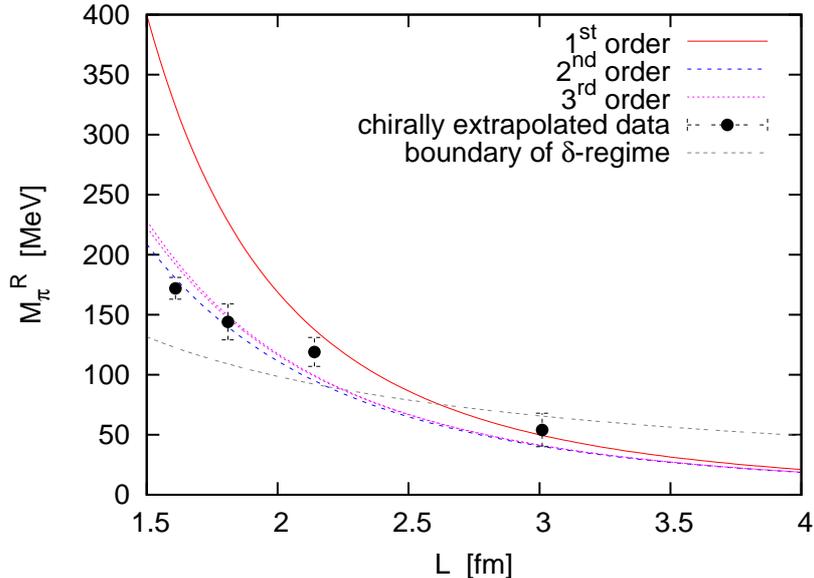
Figure 6. A comparison of our extrapolated values for the residual pion mass M_π^R with the predictions by the δ-expansion (LO, NLO, NNLO), as functions of the spatial box size L. We observe a remarkably good agreement. In our largest volume, the extrapolated M_π^R is located below the physical pion mass, and in the region $M_\pi^R L < 1$, which can be viewed as the domain of the δ-regime.

a regime where the basis of the δ-expansion (frozen spatial degrees of freedom) does not hold. This suggests that the extrapolation formula (15) applies well in a sizable domain.

5. Evaluation of Low Energy Constants

By fitting our pion mass data according to eq. (15), we obtained results for the four free fitting parameters. In Section 4 we discussed the results for M_π^R that we obtained in this way. Moreover, the results for C_1, C_2 and C_3 can be used to evaluate the notorious sub-leading LEC $\bar{l}_3$ [14]. This is how we obtained our value that we anticipated in Section 4,

$$\bar{l}_3 = 32\pi^2 F_\pi^2 \frac{C_2}{C_1} \ln \frac{C_1}{C_3 M_\pi^{\text{phys 2}}} = 4.2 \pm 0.2 \ . \tag{16}$$

(The error given here emerges from the fits, the additional systematic error would be hard to estimate). In view of other results in the literature, this value is in the upper region. An overview has been presented in Ref. [17], in particular in Table 11 and Figure 9. That overview estimates the world average as $\bar{l}_3 = 3.3(7)$.

In the previous considerations of Sections 4 and 5, we inserted the phenomenological value of the pion decay constant, $F_\pi = 92.4$ MeV. Alternatively, we could also treat F_π as a free parameter to be determined by the fits. In particular, matching M_π^R this yields in the chiral limit [14]

$$F_\pi^{\text{numerical}}\Big|_{m_q=0} = 78^{+14}_{-10} \text{ MeV} \ , \tag{17}$$

which seems a bit low. However, effective field theory considerations suggest that the value of F_π in the chiral limit should indeed be below the physical value; Ref. [17] estimates $F_\pi|_{m_q=0} \approx 86$ MeV.

6. Conclusions

The δ-regime refers to a system of pions in a finite box, typically of the shape $L^3 \times T$ with $L \lesssim M_\pi^{-1} \ll T$. It can be treated by Chiral Perturbation Theory with suitable counting rules, the δ-expansion. In particular this predicts the residual pion mass in the chiral limit, M_π^R, which has been computed recently to NNLO [9].

First attempts to simulate 2-flavour QCD in the δ-regime were confronted with technical difficulties, in particular thermalisation problems. Nevertheless we obtained good results for the pion mass with light quarks on a small lattice of size $8^3 \times 16$, with $L \simeq (0.82 \ldots 0.91)$ fm.

In another pilot study we measured pion masses in the p-regime (up to the transition region) and extrapolated them towards the δ-regime. This yields numerical results for M_π^R, which agree remarkably well with the predictions by the δ-expansion. That expansion is based on assumptions, which do not hold in the regime where the data were obtained. Therefore it would not have been obvious to predict the observed agreement. This comparison can be viewed as a numerical experiment, which led to an interesting observation.

Our pion mass fits from the p- towards the δ-regime also fix some further constants, which allow for a new determination of the (mysterious) sub-leading Low Energy Constant $\bar{l}_3$; we obtained $\bar{l}_3 = 4.2(2)$. This is somewhat above the average of the phenomenological and numerical estimates in the literature. The constants $\bar{l}_1, \ldots, \bar{l}_4$ are relevant in the effective description of $\pi\pi$ scattering. Finally we also considered the option to treat F_π as a free parameter. In the chiral limit the fits lead to a value somewhat below the phenomenological F_π.

Robust numerical results, measured manifestly inside the δ-regime, are still outstanding.

References

[1] Weinberg S 1979 *Physica* A **96** 327. Gasser J and Leutwyler H 1984 *Ann. Phys.* **158** 142
[2] Gasser J and Leutwyler H 1987 *Phys. Lett.* B **184** 83
[3] Gasser J and Leutwyler H 1987 *Phys. Lett.* B **188** 477
[4] Giusti L, Hoelbling C, Lüscher M and Wittig H 2003 *Comput. Phys. Commun.* **153** 31
[5] Bietenholz W 2010 arXiv:1007.0285 [hep-lat]
[6] Leutwyler H 1987 *Phys. Lett.* B **189** 197
[7] Neuberger H and Ziman T 1989 *Phys. Rev.* **B 39** 2608. Hasenfratz P and Leutwyler H 1990 *Nucl. Phys.* **B 343** 241. Wiese U-J and Ying H-P 1994 *Z. Phys.* B **93** 147. Beard B B and Wiese U-J 1996 *Phys. Rev. Lett.* **77** 5130. Gerber U, Hofmann C P, Jiang F-J, Nyfeler M and Wiese U-J 2009 *J. Stat. Mech.* P03021. Hofmann C P 2010 *Phys. Rev.* **B 81** 014416. Jiang F-J and Wiese U-J 2010 arXiv:1011.6205 [cond-mat.str-el]
[8] Hasenfratz P and Niedermayer F 1993 *Z. Phys.* B **92** 91
[9] Hasenfratz P 2010 *Nucl. Phys.* B **828** 201
[10] Niedermayer F and Weiermann C 2011 *Nucl. Phys.* B **842** 248
[11] Weingart M 2010 arXiv:1006.5076 [hep-lat]
[12] Hasenfratz A, Hasenfratz P, Hierl D, Niedermayer F and Schäfer A 2006 *PoS(LAT2006)178*
[13] Colangelo G, Gasser J and Leutwyler H 2001 *Nucl. Phys.* B **603** 125
[14] Bietenholz W, Göckeler M, Horsley R, Nakamura Y, Pleiter D, Rakow P E L, Schierholz G and Zanotti J M (QCDSF Collaboration) 2010 *Phys. Lett.* B **687** 410
[15] Lüscher M 1986 *Commun. Math. Phys.* **104** 177, 1986 *Commun. Math. Phys.* **105** 153
[16] Colangelo G, Dürr S and Haefeli C 2005 *Nucl. Phys.* B **721** 136
[17] FLAG Working Group of FLAVIAT (Colangelo G *et al.*) 2010 arXiv:1011.4408 [hep-lat]