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Edge Disjoint Steiner Trees in
Graphs without Large Bridges

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Abstract

A set $A$ of vertices of an undirected graph $G$ is called $k$-edge-connected in $G$ if for all pairs of distinct vertices $a, b \in A$ there exist $k$ edge disjoint $a, b$-paths in $G$. An $A$-tree is a subtree of $G$ containing $A$, and an $A$-bridge is a subgraph $B$ of $G$ which is either formed by a single edge with both end vertices in $A$ or formed by the set of edges incident with the vertices of some component of $G - A$.

It is proved that (i) if $A$ is $k \cdot (\ell + 2)$-edge-connected in $G$ and every $A$-bridge has at most $\ell$ vertices in $V(G) - A$ or at most $\ell + 2$ vertices in $A$ then there exist $k$ edge disjoint $A$-trees, and that (ii) if $A$ is $k$-edge-connected in $G$ and $B$ is an $A$-bridge such that $B$ is a tree and every vertex in $V(B) - A$ has degree 3 then either $A$ is $k$-edge-connected in $G - e$ for some $e \in E(B)$ or $A$ is $(k - 1)$-edge-connected in $G - E(B)$.

AMS subject classification: 05C70, 05C40.

Keywords: Steiner tree, packing, connectivity, bridge.

1 Introduction

All graphs considered here are supposed to be finite and undirected and may contain loops or multiple edges. For terminology not defined here see [2]. A set $A$ of vertices of a graph $G$ is called $k$-edge-connected in $G$ if for all pairs of distinct vertices $a, b \in A$ there exist $k$ edge disjoint $a, b$-paths in $G$. A Steiner tree with respect to $A$ or, briefly, an $A$-tree is a subtree of $G$ covering $A$.

By Tutte’s and Nash-Williams’s base packing theorem for graphs [14, 13] it follows readily that every $2k$-edge-connected graph has a collection of $k$ edge disjoint spanning trees (cf. [2]). It has been conjectured in [6] that there is the following generalization to $A$-trees (see also [4] and [3]).

Throughout this paper, the empty graph is considered to be a $\emptyset$-tree, and “$k$ edge disjoint $A$-trees” actually means “a family of $k$ edge disjoint $A$-trees”, so that for $|A| \leq 1$ there exist families of edge disjoint $A$-trees of any required size. It is not difficult to prove that for each $\ell$ there exists an $f_\ell(k)$ such that every $f_\ell(k)$-edge-connected set $A$ with $|A| \leq \ell$ in some graph $G$ admits a set of $k$ edge disjoint $A$-trees. The $f_\ell(k)$ derived in [6] is linear in $k$ but exponential in $\ell$, whereas from the results in [4] one can obtain a bound which is linear in both $\ell$ and $k$, with a good constant. The optimal $f_\ell$, which is $f_\ell(k) = k$, is an immediate consequence of the definitions, whereas determining the optimal $f_3$, which is $f_3(k) = \lfloor \frac{8k+3}{6} \rfloor$, turned out to be more tedious (see [6] and [4]). In both [6] and [4], conjectures on the optimum value of $f_\ell(k)$ have been made, and from the estimations in [4] it follows that Conjecture 1 is true for $|A| \leq 5$. Similar results hold if $A := V(G) - A$ is bounded: Every $(2k + 2\ell)$-edge-connected set $A$ with $|V(G) - A| \leq \ell$ in some graph $G$ admits a set of $k$ edge disjoint $A$-trees.

Recently, Lau proved that every $26k$-edge-connected set $A$ of vertices of some graph $G$ admits $k$ edge disjoint $A$-trees [7], and a bound of $24k$ is given in his thesis [8, Theorem 3.1.2]. These are the first bounds $f(k)$ which do not involve the size of $A$. Moreover, Lau’s proof yields a polytime approximation algorithm for the Steiner tree packing problem, that is, given $G$ and $A \subseteq V(G)$ with $|A| \geq 2$, find a largest set of edge disjoint $A$-trees.

If $V(G) - A$ is independent or, equivalently, $A$ is dominating in $G$, then there is a much better bound, namely $f(k) = 3k$ [3], and if every vertex in $V(G) - A$ has an even degree then $f(k) = 2k$ suffices [6], as conjectured.

First we prove a result which involves more structure of the instance $(G,A)$. An $A$-bridge is a subgraph $B$ of $G$ which is either formed by a single edge with both end vertices in $A$ or formed by the set of edges incident with the vertices of some component of $G - A$. We prove that for all integers $k, \ell \geq 0$ every $k \cdot (\ell + 2)$-edge-connected set $A$ of vertices in a graph $G$ such that every $A$-bridge has at most $\ell$ vertices in $V(G) - A$ or at most $\ell + 2$ vertices in $A$ admits a set of $k$ edge disjoint $A$-trees.

This generalizes the initially mentioned statement on spanning trees in $2k$-edge-connected graphs (set $\ell = 0$) as well as the situation that $V(G) - A$ is independent, where we have to take $\ell = 1$ and obtain the same bound $3k$ as in [3]. (It also equips us with an $f_\ell(k)$ as above, which is of the same order as the bound from [4], but with a larger constant.)

In the second part of the paper we prove that if $A$ is a $k$-edge-connected set of vertices in $G$ and $B$ is an $A$-bridge such that $B$ is a tree and every vertex in $V(B) - A$ has degree 3 then either $A$ is $k$-edge-connected in $G - e$ for some $e \in E(B)$ or $A$ is $(k - 1)$-edge-connected in $G - E(B)$. This provides a short
cut for the determination of $f_3(k)$ as in [6], and we show how one would need to generalize it in order to obtain an alternative proof of the statement of the penultimate paragraph.

2 Essential edges in cubic bridges

Given two distinct vertices $a, b$ of a graph $G$, let us denote by $\lambda_G(a, b)$ the maximum number of edge disjoint $a,b$-paths in $G$. We extend this to a mapping $\lambda_G : V(G) \times V(G) \to \mathbb{N} \cup \{0, \infty\}$ by setting $\lambda_G(a, a) := +\infty$ and extend the natural order on $\mathbb{N} \cup \{0, \infty\}$ by defining $a \leq +\infty$ for all $a \in \mathbb{N} \cup \{0, \infty\}$. An $a, b$-cut is a set of edges in $G$ which intersects the edge set of every $a, b$-path. For $A \subseteq V(G)$, an $A$-cut is an $a, b$-cut for some vertices $a \neq b$ from $A$. A variant of Menger’s theorem states that for $a \neq b$ in $V(G)$, $\lambda_G(a, b)$ equals the minimum cardinality of an $a, b$-cut (cf. [2]), so that $A \subseteq V(G)$ with $|A| \geq 2$ is $k$-edge-connected if and only if there is no $A$-cut of cardinality less than $k$. We call an edge $e = xy$ of $G$ essential (for $A$ being $k$-edge-connected in $G$), if $A$ is $k$-edge-connected in $G$ and $A$ is not $k$-edge-connected in $G - e$. This is equivalent to the statement that $A$ is $k$-edge-connected in $G$ and $e$ is contained in some $A$-cut $S$ of cardinality $k$; it is easy to see that in this case each component of $G - S$ which contains one of $x, y$ must intersect $A$. $A$ is minimally $k$-edge-connected in $G$ if every edge is essential for $A$ being $k$-edge-connected in $G$.

We start with a useful observation whose ancestors can be found in [9] and [11].

For a graph $G$ and $X, Y \subseteq V(G)$, let $E_G(X, Y)$ denote the set of edges $xy$ with $x \in X$, $y \in Y$. (If an edge $e$ is denoted by a word $xy$ then its endvertices are assumed to be $x$ and $y$.)

**Lemma 1** Let $A$ be a $k$-edge-connected set of vertices in a graph $G$. Let $y \in V(G) - A$ be a vertex of degree 3, let $xy, yz, wy$ be the three edges incident with $y$, let $S, T$ be $A$-cuts of cardinality $k$ containing $xy, yz$, respectively, let $C$ be the component of $G - S$ containing $x$, and let $D$ be the component of $G - T$ containing $y$.

Then $C \cap A \subset D \cap A$ or $A$ is $k$-edge-connected in $G - wy$.

**Proof.** Let $B_i := C \cap D$, $B_2 := C \cap D$, $B_3 := C \cap D$, $B_4 := C \cap D$, $A_i := B_i \cap A$ for $i \in \{1, 2, 3, 4\}$, and $R_{ij} := E_G(B_i, B_j)$ for $i \neq j$ in $\{1, 2, 3, 4\}$. Then $S = R_{13} \cup R_{14} \cup R_{23} \cup R_{24}$ and $T = R_{12} \cup R_{14} \cup R_{32} \cup R_{34}$. Set $Q_i = E_G(B_i, B_i) = \bigcup_{j \in \{1, 2, 3, 4\} - \{i\}} R_{ij}$ for $i \in \{1, 2, 3, 4\}$. So $xy, yz \in Q_3$. Observe that $xy \notin T$ and $wy \notin T$, for otherwise $(T - \{yz, xy\}) \cup \{wy\}$ or $(T - \{yz, wy\}) \cup \{xy\}$ would be an $A$-cut, which contradicts the connectivity condition to $A$; in particular, $w, x \in D$, so $xy \in Q_1$. Symmetrically, $w, z \in C$ (and $yz \in Q_4$), so $w \in B_3$, and the objects are located as depicted in Figure 1.

If $A_2 \neq \emptyset \neq A_3$ then $Q_2, Q_3$ are $A$-cuts, so $|Q_2|, |Q_3| \geq k$. From $|Q_2| + |Q_3| \\ \leq
It follows that one of \( A_2, A_3 \) is empty. As \( S,T \) are \( A \)-cuts, each of \( C, \overline{C}, D, \overline{D} \) intersects \( A \), implying \( A_1 \neq \emptyset \neq A_4 \). Now \( |Q_1|, |Q_4| \geq k \), and from \( |Q_1| + |Q_4| \leq |R_{12}| + |R_{13}| + |R_{14}| + |R_{41}| + |R_{42}| + |R_{43}| = |S| + |T| - 2|R_{14}| \leq 2k \) we deduce \( |Q_2| = |Q_3| = k \), implying that \( (Q_3 - \{xy, yz\}) \cup \{wy\} \) is an \( A \)-cut, again a contradiction.

It follows that one of \( A_2, A_3 \) is empty. As \( S,T \) are \( A \)-cuts, each of \( C, \overline{C}, D, \overline{D} \) intersects \( A \), implying \( A_1 \neq \emptyset \neq A_4 \). Now \( |Q_1|, |Q_4| \geq k \), and from \( |Q_1| + |Q_4| \leq |R_{12}| + |R_{13}| + |R_{14}| + |R_{41}| + |R_{42}| + |R_{43}| = |S| + |T| - 2|R_{14}| \leq 2k \) we deduce \( |Q_2| = |Q_3| = k \), implying that \( (Q_3 - \{xy, yz\}) \cup \{wy\} \) is an \( A \)-cut, again a contradiction.

Figure 1: Location of the objects in Lemma 1.

\(|R_{21}| + |R_{23}| + |R_{24}| + |R_{31}| + |R_{32}| + |R_{34}| = |S| + |T| - 2|R_{14}| \leq 2k\) we deduce \( |Q_2| = |Q_3| = k \), implying that \((Q_3 - \{xy, yz\}) \cup \{wy\}\) is an \( A \)-cut, again a contradiction.

It follows that one of \( A_2, A_3 \) is empty. As \( S,T \) are \( A \)-cuts, each of \( C, \overline{C}, D, \overline{D} \) intersects \( A \), implying \( A_1 \neq \emptyset \neq A_4 \). Now \( |Q_1|, |Q_4| \geq k \), and from \( |Q_1| + |Q_4| \leq |R_{12}| + |R_{13}| + |R_{14}| + |R_{41}| + |R_{42}| + |R_{43}| = |S| + |T| - 2|R_{14}| \leq 2k \) we deduce \( |Q_2| = |Q_3| = k \), implying that \((Q_3 - \{xy, yz\}) \cup \{wy\}\) is an \( A \)-cut, again a contradiction.

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If \( A_3 = \emptyset \) then \( A \) is \( k \)-edge-connected in \( G - wy \), and if \( A_3 \neq \emptyset \) then \( A_2 = \emptyset \) and \( C \cap A = A_1 \subset A_1 \cup A_3 = D \cap A \).

Lemma 2 Let \( A \) be a \( k \)-edge-connected set of vertices in a graph \( G \).

Let \( P = v_0, v_1, \ldots, v_\ell \) be a path of length \( \ell \geq 2 \) such that for \( i \in \{1, \ldots, \ell - 1\} \), \( v_i \) is a vertex in \( V(G) - A \) of degree 3 and the three edges incident with \( v_i \) are essential for \( A \) being \( k \)-edge-connected. Let \( S, T \) be \( A \)-cuts of cardinality \( k \) containing \( v_0v_1, v_{\ell - 1}v_\ell \) respectively, let \( C \) be the component of \( G - S \) containing \( v_0 \) and \( D \) be the component of \( G - T \) containing \( v_{\ell - 1} \).

Then \( C \cap A \subset D \cap A \).
Figure 2: Necessity of the degree condition in Theorem 1.

Proof. For $i \in \{2, \ldots, \ell - 1\}$, let $T_i$ be a cut of cardinality $k$ containing $v_{i-1}v_i$. Set $T_1 := S$ and $T_\ell := T$. Let $C_i$ be the component of $G - T_i$ containing $v_{i-1}$, so $C_1 = C$ and $C_\ell = D$. By Lemma 1, applied to $v_{i-1}, v_i, v_{i+1}, T_i, T_{i+1}$ for $x, y, z$, $S, T$ we deduce $C_i \cap A \subset C_{i+1} \cap A$. □

Lemma 2 enables us to reduce cycles in “internally 3-regular” $A$-bridges, yielding the main result of this section.

**Theorem 1** Let $A$ be a $k$-edge-connected set of vertices in a graph $G$ and let $B$ be an $A$-bridge such that every vertex $x \in V(B) - A$ has degree 3 in $G$ and the three edges incident with $x$ are essential for $A$ being $k$-edge-connected.

Then every $A$-cut of cardinality $k$ contains at most one edge of $B$. In particular, $B$ is a tree whose end vertices are the vertices of $V(B) \cap A$.

Proof. Suppose, to the contrary, that $S$ is an $A$-cut of cardinality $k$ containing $e \neq f$ in $E(B)$. Then we may choose a path $P = v_0, v_1, \ldots, v_\ell$ of length $\ell \geq 2$ with $v_i \in V(B) - A$ for all $i \in \{1, \ldots, \ell - 1\}$ such that $E(P) \cap S = \{v_0v_1, v_{\ell-1}v_\ell\}$. Let $C$ be the component of $G - S$ containing $v_0$ and let $D$ be the component of $G - S$ containing $v_{\ell-1}$. By Lemma 2, applied to $T = S$, $C \cap A \subset D \cap A$. However, $D = \overline{C}$ by choice of $P$, a contradiction.

As every cycle intersects every cut in an even number of edges and every edge of $B$ is contained in some $A$-cut of cardinality $k$, $B$ is a tree, and, as no vertex of $V(B) \cap A$ separates the bridge $B$, the second part of the statement follows. □

The degree condition to $V(B) - A$ in Theorem 1 is necessary, as the graph $G$ in Figure 2 shows. The vertices in $A$ are colored black, and $A$ is minimally 2-edge-connected in $G$. $G$ itself is the unique $A$-bridge, and every $A$-cut must intersect it twice.

3 Graphs without large binary bridges

We proceed with a consequence of the following Theorem from [3]. Basic hypergraph terminology can be found in [1]. A hypergraph is called $k$-partition-connected for some integer $k \geq 0$ if

$$e_G(P) \geq k \cdot (|P| - 1)$$  \hspace{1cm} (1)
holds for every partition \( P \) of \( V(G) \), where \( e_G(P) \) denotes the number of edges of \( G \) which intersect at least two distinct members of \( P \). Observe that every 1-partition-connected hypergraph is connected.

**Theorem 2** [3] A hypergraph is \( k \)-partition-connected if and only if it has \( k \) edge disjoint 1-partition-connected spanning subhypergraphs.

From Theorem 2 we deduce the following.

**Theorem 3** Let \( r \geq 2 \) and \( A \) be an \( rk \)-edge-connected set of vertices in some graph \( G \) such that \( X := V(G) - A \) is independent in \( G \) and \( d_G(x) \leq r \) for every \( x \in X \).

Then there exist \( k \) edge disjoint \( A \)-trees in \( G \) which are pairwise disjoint on \( X \).

**Proof.** Without loss of generality we may assume that \( A \) is independent in \( G \), for subdividing every edge in \( E(G(A)) \) once and adding the subdivision vertices to \( X \) keeps the conditions to the new instances \( G', X', A' = A \) alive, and if \( G' \) admits a set of \( k \) edge disjoint \( A \)-trees pairwise disjoint on \( X' \) then we may construct easily a system of \( k \) edge disjoint \( A \)-trees of \( G \) disjoint on \( X \).

The set family \( (e_x := N_G(x))_{x \in X} \) constitutes a hypergraph \( H \) on \( A \). Let \( P \) be a partition of \( V(H) = A \) into at least two classes. Let \( Y \) denote the set of vertices in \( X \) which have neighbors in at least two members of \( P \), and for \( P \in P \), let \( a(P) \) denote the number of edges in \( G \) which connect a vertex in \( P \) to some vertex in \( Y \). Since \( A \) is \( rk \)-edge-connected in \( G \), \( a(P) \geq rk \) for all \( P \in P \), and since \( d_G(x) \leq r \) for all \( x \in X \) we deduce \( r \cdot e_H(P) \geq \sum_{x \in Y} d_G(x) = \sum_{P \in P} a(P) \geq rk|P| \), so (1) holds.

By Theorem 2, \( H \) admits \( k \) edge disjoint (1-partition-) connected spanning subhypergraphs \( H_1, \ldots, H_k \). Let \( X_i := \{x \in X : e_x \in E(H_i)\} \). Then the graphs \( G(X_i \cup A) \), \( i \in \{1, \ldots, k\} \) are connected subgraphs and pairwise disjoint on \( X \). Choose a spanning tree of each \( G(X_i \cup A) \). This produces \( k \) edge disjoint \( A \)-trees in \( G \) disjoint on \( X \). \( \square \)

The basic reduction technique to prove the following result is to split pairs of edges at some vertex in a graph \( G \). A splitting at \( x \) is a pair \( p = (wx, xy) \) of distinct edges. The graph \( G(wx, xy) = G(p) \) obtained from \( G - \{wx, xy\} \) by adding a single new bypass edge from \( w \) to \( y \) is also said to be obtained from \( G \) by performing \( p \). A splitting \( p \) at \( x \) is admissible if \( \lambda_{G(p)}(a,b) = \lambda_G(a,b) \) for all \( a, b \in V(G) - \{x\} \). Mader’s Splitting Lemma [10, 12] can be stated as follows.

**Lemma 3** [10, 12] If \( x \) is a nonseparating vertex of the graph \( G \) of degree distinct from 0, 1, 3 then there exists an admissible splitting at \( x \).

Now we are prepared to prove the main result of this section.
Theorem 4 Let $\ell \geq 0$ and $A$ be a $k \cdot (\ell + 2)$-edge-connected set of vertices in some graph $G$ such that every $A$-bridge has at most $\ell$ vertices in $V(G) - A$ or at most $\ell + 2$ vertices in $A$.

Then there exist $k$ edge disjoint $A$-trees.

Proof. We prove this by induction on $|E(G)| + |V(G)|$. If there is an admissible splitting $p$ at some vertex $x \in V(G) - A$ then $A$ is $k \cdot (\ell + 2)$-edge-connected in $G(p)$, and the vertex set of every $A$-bridge in $G(p)$ is contained in the vertex set of some $A$-bridge of $G$; by induction, $G(p)$ has $k$ edge disjoint $A$-trees, and from these one easily obtains $k$ edge disjoint $A$-trees in $G$. Hence there is no such splitting and, by Lemma 3 we may assume that every vertex in $V(G) - A$ either separates $G$ or has degree 0, 1 or 3. If $x \in V(G) - A$ has degree 0 or 1 then we apply induction to $G - x$ straightforwardly.

Now suppose that $x \in V(G) - A$ separates $G$ and let $\mathcal{C}$ be the set of components of $G - x$. If there is a $C \in \mathcal{C}$ not containing vertices from $A$ then we apply induction to $G - C$ straightforwardly. Otherwise, we take any $C \in \mathcal{C}$ and observe that for any $a \in A \cap C \neq \emptyset$ and any $b \in A - C \neq \emptyset$ there exist $k \cdot (\ell + 2)$ edge disjoint $a,b$-paths; since each of them contains $x$, $A' := (A \cap C) \cup \{x\}$ is $k \cdot (\ell + 2)$-edge-connected in $G' := G(C \cup \{x\})$. Since every $A'$-bridge $B'$ of $G'$ is a subgraph of some $A$-bridge of $G$ which, moreover, contains at least one vertex from $A - A'$ if $B'$ contains $x$, we may apply induction to obtain $k$ edge disjoint $A'$-trees $T_{C,1}, \ldots, T_{C,k}$ in $G'$ — and so $(\bigcup_{C \in \mathcal{C}} T_{C,1})_{i \in \{1,\ldots,k\}}$ is the desired family of edge disjoint $A$-trees.

Hence every vertex in $V(G) - A$ has degree 3. Furthermore, we may assume that every edge $e$ is essential for $A$ being $k \cdot (\ell + 2)$-edge-connected in $G$, as otherwise $A$ is $k \cdot (\ell + 2)$-edge-connected in $G - e$ and the statement follows inductively. By Theorem 1, every $A$-bridge is a tree such that its vertices from $A$ have degree 1 and its vertices from $V(G) - A$ have degree 3. As the number of end vertices of such a tree equals 2 plus the number of its non-end-vertices, the conditions to the $A$-bridges imply that it has at most $\ell + 2$ end vertices.

Let $G'$ be obtained from $G$ by contracting each component of $G - A$ to a single vertex. $A$ remains $k \cdot (\ell + 2)$-connected in $G'$, $X := V(G') - A$ is independent, and $d_{G'}(x) \leq \ell + 2$ for every $x \in X$. By Theorem 3, there exist $k$ edge disjoint $A$-trees in $G'$ which are disjoint on $X$, and from these one easily obtains $k$ edge disjoint $A$-trees in $G$. □

For $\ell = 0$, Theorem 4 states that every $2k$-edge-connected graph admits $k$ edge disjoint spanning trees. For $\ell = 1$ we deduce the existence of $k$ edge disjoint $A$-trees if $A$ is $3k$-edge-connected and $V(G) - A$ is independent, which was a Corollary in [3]. Equivalently, one could say that every $3k$-edge-connected dominating set $A$ admits $k$ edge disjoint $A$-spanning trees.

Furthermore, if $|A|$ is bounded from above by some $\ell$ and is $k \cdot \ell$-edge-connected then $G$ admits $k$ edge disjoint $A$-trees, as every $A$-bridge contains at most $\ell$
vertices from $A$. This bound has the same order of magnitude than the one in [4], but a larger constant.

In [5] it as been shown that if $A$ is minimally $k$-edge-connected in $G$ and every vertex in $V(G)\setminus A$ has odd degree then $|V(G)| \leq (k+1)|A| - 2k$ [5, Theorem 6]. Let me briefly sketch an alternative proof, relying on Theorem 1 and the methods of the preceding proof. We perform induction on $|V(G)|$. By performing admissible splittings at vertices from $V(G)\setminus A$ we can achieve that every vertex in $V(G)\setminus A$ has degree 3, as these splittings keep $A$ minimally $k$-edge-connected in $G$. Now every bridge is binary by Theorem 1, which implies that for each $b \in A$, there are $d_G(b)$ edge disjoint $b,A \setminus \{b\}$-paths, each in a distinct $A$-bridge. If $d_G(b) \geq k + 2$ we thus may perform an admissible splitting at $b$ keeping $A$ minimally $k$-edge-connected. If $d_G(b) = k + 1$ then we perform an admissible splitting at $b$, which keeps $A \setminus \{b\}$ minimally $k$-edge-connected, and consider the bypass edge $h$ of the splitting. Subdividing $h$ by a new vertex $y$ and adding a new edge from $y$ to $b$ produces a new graph in which $A$ is minimally $k$-edge-connected. Hence we may transform the instance to a new one with possibly more vertices where every vertex in $A$ has degree $k$ and every $A$-bridge is binary. Now consider the $A$-bridges in $G$, say $B_1, \ldots, B_\ell$. Then $\ell \geq k$, and so $|V(G)\setminus A| = \sum_{i=1}^{\ell} |V(B_i)\setminus A| = \sum_{i=1}^{\ell} |V(B_i)\cap A| - 2 \ell \leq k \cdot |A| - 2k$, which implies the statement.

4 Removing a single binary bridge

Let $G$ be a graph and $A \subseteq V(G)$. Let us call an $A$-bridge $B$ binary if $B$ is a tree and the vertices in $V(B)\setminus A$ have degree 3 (those of $V(B)\cap A$ must have degree 1 since $B$ is both an $A$-bridge and a tree). Hence Theorem 1 implies that if every edge of $B$ is essential for $A$ being $k$-edge-connected and every vertex in $V(B)\setminus A$ has degree 3 then $B$ is binary. In this section we prove that if every edge of a binary bridge is essential for $A$ being $k$-edge-connected in $G$ then $G - E(B)$ is $(k - 1)$-edge-connected. This is far from being true for arbitrary $A$-bridges: They might disconnect $A$ although each of its edges is essential, as it is shown by replacing every edge $xy$ of a tree on at least 3 vertices whose end vertices constitute $A$ with $k$ distinct edges connecting $x,y$ (Figure 2 displays the case where the tree is a path of length 2 and $k = 2$).

We prefix the following lemma.

**Lemma 4** Let $A$ be a set of vertices in some graph $G$, $B$ be an $A$-bridge, and $x,y \in V(G) - (A \cup V(B))$.

If $A$ is $k$-edge-connected in $G - E(B)$ and there exist $k$ edge disjoint $x,y$-paths in $G$ then there exist $k$ edge disjoint $x,y$-paths in $G - E(B)$.
Proof. For suppose, to the contrary, that there exists an \(\{x, y\}\)-cut \(T\) in \(G - E(B)\) with \(|T| < k\), and let \(C\) be the component of \((G - E(B)) - T\) containing \(x\). Then \(y \in C\). As \(A\) is \(k\)-edge-connected in \(G - E(B)\), \(A \subseteq C\) or \(A \subseteq C^c\), and we may assume by symmetry that \(A \subseteq C\). However, there exist \(k\) edge disjoint \(x, y\)-paths in \(G\), and since \(x \notin A \cup V(B)\), each of them contains an \(x, A \cup \{y\}\)-path in \(G - E(B)\), which must intersect \(T\) — a contradiction. \(\square\)

**Theorem 5** Let \(A\) be a \(k\)-edge-connected set of vertices of some graph \(G\) and suppose that \(B\) is a binary \(A\)-bridge such that every edge of \(B\) is essential for \(A\) being \(k\)-edge-connected.

Then \(A\) is \((k - 1)\)-edge-connected in \(G - E(B)\).

**Proof.** For an edge \(xy \in E(B)\), let \(T(x, y)\) denote the set of \(A\)-cuts of cardinality \(k\) which contain \(xy\). Then \(T(x, y) \neq \emptyset\), since \(xy\) is essential. For \(T \in T(x, y)\), let \(C(x, y, T)\) denote the component of \(G - T\) containing \(x\), and let \(C(x, y) := \{C(x, y, T) : T \in T(x, y)\}\). Furthermore, let \(A(x, y)\) denote the set of endvertices in the component \(B(x, y)\) of \(B - xy\) which contains \(x\). As the intersection of any \(T \in T(x, y)\) with \(E(B)\) equals \(\{xy\}\) by Theorem 1, \(C(x, y, T) \cap V(B) \cap A = A(x, y)\). It is well-known and easy to see that \(C(x, y)\) is closed under intersection, hence there is a unique minimal element in \(C(x, y)\) with respect to \(\subseteq\), namely \(C(x, y) := \bigcap C(x, y)\). Let \(T(x, y) := E_G(C(x, y), C(x, y))\) denote the corresponding \(A\)-cut from \(T(x, y)\).

Let \(\ell := \lceil (k - 1)/2 \rceil\).

**Claim 1.** For each \(xy \in E(B)\), \(A(x, y)\) is \(\ell\)-edge-connected in \(C(x, y) - E(B)\).

We perform induction on \(|A(x, y)|\). The statement is trivially true if \(|A(x, y)| = 1\). If \(|A(x, y)| > 1\) then \(x \in V(B) - A\) and there exist distinct \(x_1, x_2 \in C(x, y) \cap N_G(x)\). \(A(x, y)\) is the disjoint union of the two nonempty sets \(A_1 := A(x_1, x)\) and \(A_2 := A(x_2, x)\). Since \(A_1 \subseteq D_1 := C(x, y) \cap C(x_1, x)\) and \(A_2 \subseteq C(x, y) \cap \overline{C(x, x)}\) we deduce that \(E_G(D_1, D_2)\) is an \(A\)-cut of cardinality \(k\) (similar to the argument in the proof of Lemma 1). Since \(x_1 \in D_1\) and \(x \in D_1\), \(D_i \subseteq C(x_1, x)\) follows — so \(D_i = C(x_i, x)\) by minimality of \(C(x_i, y)\). Therefore, \(C(x_1, x) \cap C(x, y) = D_i \cap C(x, y) = \emptyset\) for \(i \in \{1, 2\}\). By induction, each of \(A_1, A_2\) is \(\ell\)-edge-connected in \((C(x, y) - E(B)) \cup \{(x_1, x) - E(B)\} \cup \{(x_2, x) - E(B)\}\), and it suffices to prove that there exist \(\ell\) edge disjoint \(A_1, A_2\)-paths in \((C(x, y) - E(B))\).

In \(G\), there exist \(k\) edge disjoint \(A_1, A_2\)-paths. Since the collection of their edges must cover \(T(x_1, x) \cup T(x_2, x)\), \(x_1x_2\) is a subpath of one of them, and \(xy\) is contained in neither of them. It follows that there exists a set \(\mathcal{P}\) of \(k - 1\) \(A_1, A_2\)-paths in \(G - (E(B(x, y)) \cup \{xy\})\). Since every path in \(\mathcal{P}\) which intersects \(C(x, y)\) must contain two edges in \(T(x, y) - \{xy\}\), there are at most \(\lfloor (k - 1)/2 \rfloor\) such paths. Consequently, \(\mathcal{P}\) contains at least \(|\mathcal{P}| - \lfloor (k - 1)/2 \rfloor = \ell\) \(A_1, A_2\)-paths in \(C(x, y)\) not intersecting \(E(B)\), which proves Claim 1.

**Claim 2.** For each \(xy \in E(B)\), \(A(x, y)\) is \((k - 1)\)-edge-connected in \(G - E(B)\).
Again, the statement is trivially true for \(|A(x, y)| = 1\). Again, if \(|A(x, y)| > 1\) then \(x \in V(B) - A\), there exist distinct \(x_1, x_2 \in C(x, y) \cap N_G(x)\), and \(C(x_1, x)\) and \(C(x, y)\) are disjoint for \(i \in \{1, 2\}\); by induction, each of \(A_1 := A(x_1, x)\) and \(A_2 := A(x_2, x)\) is \((k - 1)\)-edge-connected in \(G - E(B)\), and it suffices to prove that there exist \(k - 1\) edge disjoint \(A_1, A_2\)-paths in \(G - E(B)\).

As in the proof of Claim 1, we find a set \(P\) of \(k - 1\) edge disjoint \(A_1, A_2\)-paths in \(G - (E(B(x, y)) \cup \{xy\})\). We consider these paths as being oriented from \(A_1\) towards \(A_2\).

For \(P \in P\), let \(Q(P)\) be the set of oriented subpaths of \(P\) of length at least two whose endvertices are in \(C(x, y)\) and whose internal vertices are in \(C(x, y)\).

Let \(q := |\{P \in P : Q(P) \neq \emptyset\}|\), let \(R(P)\) denote the set of components of \(P - E(Q(P))\), and let \(Q := \bigcup_{P \in P} Q(P)\).

Let \(H\) be obtained from \(G(C(x, y))\) by adding two new vertices \(a, b\), taking the union with the paths from \(Q\) and redirecting their initial edges from \(a\) towards the respective old second vertices and their terminal edges from the respective penultimate vertices towards \(b\). By construction, there exist \(q\) edge disjoint \(a, b\)-paths in \(H\), and since their initial and terminal edges correspond to pairwise distinct edges in \(T(x, y) - \{xy\}\), \(q \leq \lfloor (k - 1)/2 \rfloor \leq \ell\).

Now \(B(y, x)\) is an \(A(y, x)\)-bridge in \(H\), and \(A(y, x)\) is \(q\)-edge-connected in \(H - E(B)\) by Claim 1 applied to \(yx\) for \(xy\) (since \(C(x, y) \supseteq C(y, x)\)). Applying Lemma 4 to the appropriate objects we find a set \(S\) of \(q\) edge disjoint \(a, b\)-paths in \(H - B(y, x)\), and the set of edges in \(G\) corresponding to the edges of the paths in \(S\) together with the edges of the paths in the sets \(R(P)\) form a subgraph of \(G - E(B)\) which contains \(k - 1\) edge disjoint \(A_1, A_2\)-paths.

This proves Claim 2.

**Claim 3.** \(A \cap V(B)\) is \((k - 1)\)-edge-connected in \(G - E(B)\).

Take \(u \neq v \in A \cap V(B)\). There exists an \(u, v\)-path \(P\) in \(B\), and if it has length less than 2 then \(|E(B)| = 1\) and we find \(k - 1\) edge disjoint \(u, v\)-paths in \(G - E(B)\) by trivial reasons. If, otherwise, \(P\) contains a vertex \(x\) in \(V(B) - A\) then \(x\) has a neighbor \(y \in V(B) - V(P)\), and, consequently, \(u, v \in A(x, y)\), and Claim 2 yields \(k - 1\) edge disjoint \(u, v\)-paths, proving Claim 3.

For an arbitrary \(u \in A - V(B)\), we consider \(v \in A \cap V(B)\). As there exist \(k\) edge disjoint \(u, v\)-paths in \(G\), there exist \(k\) edge disjoint \(u, A \cap V(B)\)-paths in \(G - E(B)\). Together with Claim 3 this proves the statement of the Theorem. □

Theorem 5 provides a two pages short cut in the argument of [6] showing that any \(\lfloor \frac{2k+2}{6} \rfloor\)-edge-connected set \(\{a, b, c\}\) of vertices admits \(k\) edge disjoint \(\{a, b, c\}\)-trees (that is, \(f_3(k) \leq \lfloor \frac{2k+3}{6} \rfloor\), cf. introduction): Performing induction on \(k\), we first reduce the problem to the case that every \(\{a, b, c\}\)-bridge of the given graph \(G\) is binary, just as in the proof of Theorem 4; if \(V(G) = \{a, b, c\}\) then the statement follows by a result on spanning trees from [6], and oth-
erwise there is a further vertex $x$ with edges $xa, xb, xc$ which constitute a binary \{$a, b, c\}$-bridge $B$ and, at the same time, an \{$a, b, c\}$-tree. By Theorem 5, $G - E(B)$ is $(\lfloor \frac{8k + 3}{6} \rfloor - 1)$-edge-connected and hence $\lfloor \frac{8(k-1)+3}{6} \rfloor$-edge-connected; so there are $k - 1$ edge disjoint \{$a, b, c\}$-trees in $G$ by induction, and they form, together with $B$, the desired family of $k$ edge disjoint \{$a, b, c\}$-trees of $G$.

It seems to be a difficult problem to generalize Theorem 5 to the deletion of more than one $A$-bridge. It could be possible that under the assumptions that $A$ is minimally $k$-edge-connected in $G$ and that there are only binary $A$-bridges, $A$ remains $(k - \ell)$-edge-connected in any graph obtained from $G$ by removing the edges of any $\ell$ $A$-bridges. For $k = 2$ this is true by Theorem 5, but it is not clear why for some 3-edge-connected set $A$ there cannot exist $B, B'$ such that $A$ is disconnected in $G - (E(B) \cup E(B'))$. The case $\ell = k - 1$ is particularly interesting, as it would imply that the edge connectivity of $A$ is inherited to the “bridge hypergraph” if all bridges are binary:

**Problem 1** *Is it true that if $A$ is a minimally $k$-edge-connected set of vertices in some graph $G$ and every $A$-bridge is a binary tree then the hypergraph $H$ on $A$ whose edges are formed by the family of sets of endvertices of $A$-bridges in $G$ is $k$-edge-connected?*

An affirmative answer to Problem 1 would yield an alternative proof for Theorem 4: After reduction to the case that $A$ is minimally $k \cdot (\ell + 2)$-edge-connected in $G$ and every $A$-bridge is binary, as in the present proof of Theorem 4, we would know that $H$ as in Problem 1 is $k \cdot (\ell + 2)$-edge-connected and every edge of $H$ had at most $\ell + 2$ vertices. By another result of [3], $H$ is $k$-partition-connected and thus admits $k$ (1-partition-) connected spanning subhypergraphs, which easily yield the desired family of $k$ edge disjoint $A$-trees in $G$.

**References**


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